QUARTERLY OF APPLIED MATHEMATICS

Vol. VII APRIL, 1949 No. 1

RELAXATION OSCILLATIONS[1]

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1. The equation

$$\frac{d^2x}{d\tau^2} + \mu f(x)\frac{dx}{d\tau} + x = 0 \tag{1}$$

has been studied by E. and H. Cartan [1], van der Pol [2]–[4], Liénard [5], Levinson and Smith [6] and Haag [7], among others. A large number of intriguing and interesting applications of (1) may be found in the work of van der Pol and in [8].

In this paper we study a class of relaxation oscillations related to (1) which is simpler though not included in the general equation of Levinson and Smith [6]. As has been done for van der Pol's special case, $f(x) = x^2 - 1$, by Flanders and Stoker [9], we give the explicit construction of regions $E(\mu)$ of the phase plane each of which contains a single periodic solution. We have not strived for generality nor included possible refinements of the regions but hope to illustrate the relationship between solving differential inequalities and the construction of such regions. We do abandon the assumption that $f(x)$ be even, as the asymmetric case is of practical interest.

In Sec. 2 we consider a simple physical example of the equation to be studied and anticipate some of the results by heuristic arguments. The regions are constructed in Sec. 3 and the uniqueness of the periodic solutions within each region is proved in Sec. 4. In Sec. 5 we show how bounds on periods and amplitudes and their asymptotic values can be computed.

2. Consider the differential equation

$$L\frac{d^2q}{dt'^2} + F\left(\frac{dq}{dt'}\right) + q/C = 0 \tag{2}$$

for an LRC circuit with R non-linear and non-passive. $F(i)$ is the voltage-current characteristic of R, and q is the charge. A suitable choice of units for q and t' (time)—$q = (LC)^{1/2}z$ and $t' = (LC)^{1/2}\tau$—gives Rayleigh's equation

$$\frac{d^2z}{d\tau^2} + \mu F\left(\frac{dz}{d\tau}\right) + z = 0, \qquad \text{where} \qquad \mu = (C/L)^{1/2}. \tag{3}$$

Liénard's equation (1) is equivalent to (3). It can be obtained from (3) by differentiating

[1]Received Aug. 19, 1948. This work was carried out under an Office of Naval Research Contract. The author wishes in particular to thank Professor Lefschetz for his assistance and constant encouragement.

and placing $x = dz/d\tau$ and $f(x) = F'(x)$ ($f(x)$ is the dynamic resistance of R). Conversely, (2) can be obtained from (3) by integration. Also (1) and (3) are equivalent to the systems

$$\frac{dw}{dt} = -f(x)w - \epsilon^2 x$$

$$\frac{dx}{dt} = w, \tag{4}$$

$$\frac{dx}{dt} = -F(x) - y = G(x) - y$$

$$\frac{dy}{dt} = \epsilon^2 x, \tag{5}$$

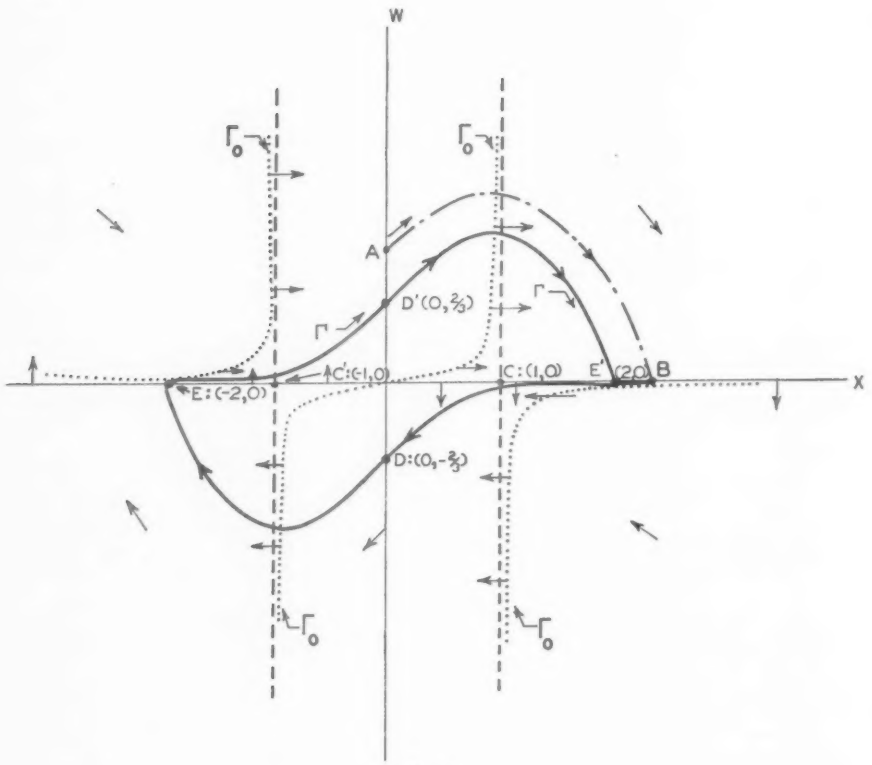

FIG. 1.

where $\epsilon = 1/\mu = (L/C)^{1/2}$, $t = \mu\tau = t'/C$, $y = \mu z = q/L$, $x = dz/d\tau = dq/dt'$, $G(x) = -F(x)$, and $f(x) = F'(x)$. In the physical problem x represents current. Its units have remained unchanged.

Although it will be more convenient to work in the (x, y) phase plane, let us first examine the behavior of the paths (solutions) in the (x, w) plane with $f(x) = x^2 - 1$, van der Pol's case. The dotted curve in Fig. 1 is the isocline Γ_0, corresponding to $dw/dt = 0$; i.e. $w = -\epsilon^2 x/f(x)$. Along the x-axis, $dx/dt = 0$, and the slopes of the paths have the general aspect indicated by the arrows. For ϵ quite small the curve Γ_0 approaches the lines $x = \pm 1$ and $w = 0$. When $(1 - x^2) \gg \epsilon^2 x$, i.e., not too close to Γ_0, then $dw/dx = (1 - x^2)$ and $w = x - x^3/3 + c$. Starting with a path through a point such as A, the path is approximately $w = x - x^3/3 + c$ until reaching the point B. From B the path is confined to move between $w = 0$ and Γ_0 until reaching C at $(1, 0)$. Here the path leaves Γ_0 and is now approximately $w = x - x^3/3 - 2/3$ from C through D until reaching $w = 0$ at E. As before, the path is practically horizontal from E to C' where it follows $w = x - x^3/3 + 2/3$ until reaching E'. Here it begins to repeat the cycle Γ through $DED'E'$. Similarly, starting inside Γ the path would approach Γ, and the limit cycle is stable. Thus we expect the amplitude of the oscillation (the maximum value of $|x|$) to approach 2 as ϵ becomes small. From E' to C, dx/dt is small and x decreases slowly from 2 to 1 and then drops rapidly from 1 to -2, giving rise to the relaxation oscillation pictured in Fig. 2.

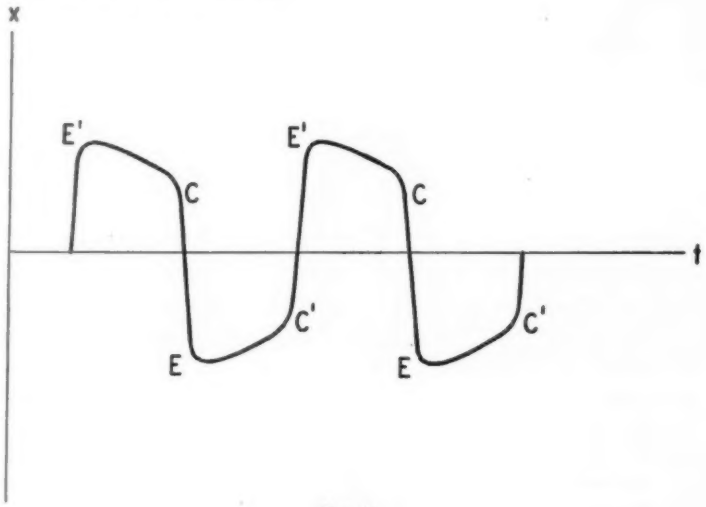

FIG. 2.

The (x, y) phase plane has the advantage that the isoclines $dx/dt = 0$ and $dy/dt = 0$ do not depend upon ϵ, and the curve $w = -\int f(x)\,dx = G(x) + c$ becomes a horizontal line. From Eqs. (5) we see that the curve

$$\Delta : y = G(x)$$

corresponds to the isocline $dx/dt = 0$, and $x = 0$ to $dy/dt = 0$. Consider a $G(x)$ such as that in Fig. 3.

In the region I (above Δ and $x > 0$) and in the region III (below Δ and $x < 0$) the paths have positive slopes. In regions II and IV the slopes are negative, and the paths encircle the origin in a counter-clockwise direction. When ϵ is very small we see from

(5) that the path through a point such as A will be almost horizontal until reaching Δ. If the slope of Δ is positive, the path will cross Δ and continue moving horizontally until approaching Δ again. At B the slope of Δ is negative and the path tends to follow Δ;

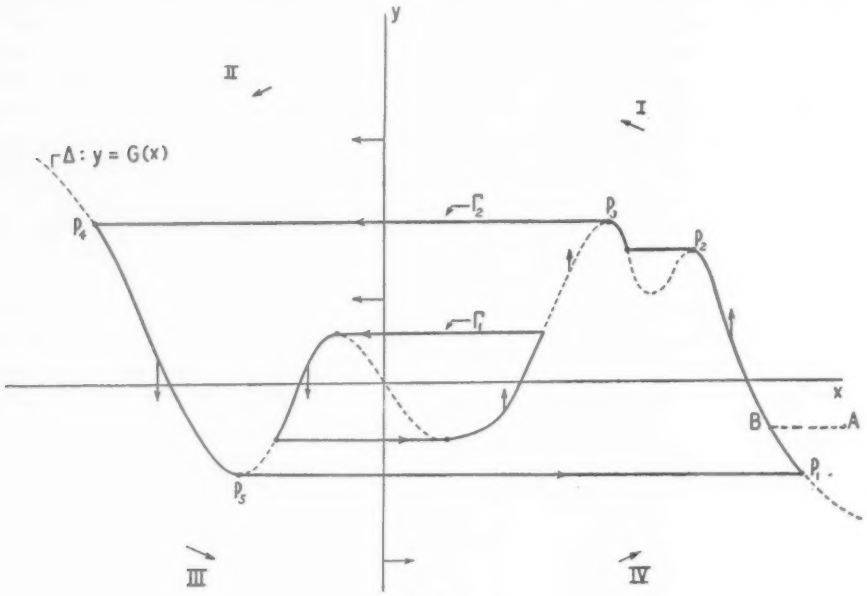

FIG. 3.

since if it moved away from Δ it would tend to move horizontally to the left and hence back towards Δ. Upon reaching a maximum point P_2, the path leaves Δ, is horizontal until returning to Δ and so on. Thus the path follows Γ_2 and we expect there to be periodic solutions (limit cycles) which lie near Γ_2 when ϵ is small. The limit cycles which approach Γ_2 would correspond to stable[2] periodic solutions. In a similar manner by letting $t \to -\infty$, i.e., moving in the direction of decreasing time, a curve such as Γ_1 is obtained and we expect it to be approached by unstable periodic solutions as $\epsilon \to 0$. The function $G(x)$ of Fig. 3 represents a case of so called "hard oscillating" conditions (see [8]). In order that the system approach a steady oscillation, it must initially be driven outside the region enclosed by Γ_1.

3. Let us make two simplifying observations. We see that Eqs. (5) are unchanged if y is replaced by $-y$, x by $-x$, and $G(-x)$ by $-G(x)$. Geometrically, this means turning our picture of the phase plane upside down (through 180°). Hence we can limit our discussion to the right half-plane $x \geq 0$, and then by this transformation apply the result to the left half-plane $x \leq 0$. Also if in (5) t is replaced by $-t$ and y by $-y$, we obtain a system of the same form with $-G(x)$ in place of $G(x)$. Thus, upon replacing $G(x)$ by $-G(x)$, in a theorem which concerns stable periodic solutions, we obtain a corresponding theorem for unstable periodic solutions.

[2]Throughout we use the term *stability* to mean "absolute orbital stability" as defined in [11]. By unstable we mean that stability is obtained if t is replaced by $-t$.

The term *enclosure* will refer to a region of the (x, y) phase plane which contains a periodic solution. To be more specific, our enclosures will be annular closed regions E (see Fig. 4) whose boundaries are a pair of simple, closed, sectionally-smooth curves C_0 and C_i, with C_i inside C_0.

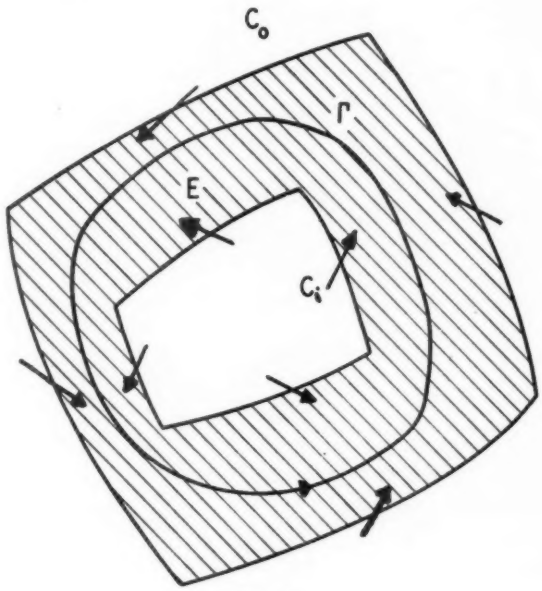

FIG. 4.

These boundaries are constructed to have the property that a path γ through a point P of the boundary enters E; i.e., if at $t = t_0$, γ is at P, then for all $t > t_0$ γ is in E. If there are no critical points in E, it follows from the Poincairé-Bendixson theorem (see, e.g., [11]), that E is an enclosure.

We assume the behavior of Fig. 5, i.e. that:

I. *$G'(x)$ is continuous for all x, and $G(0) = 0$;*

II. *there exist points $a_1 < a_2 < 0 < a_3 < a_4$ such that*

(1) *$G(a_1) = G(a_3)$ and $G(a_2) = G(a_4)$,*

(2) *$G(a_2) \leq G(x) \leq G(a_3)$ for $a_1 \leq x \leq a_4$,*

(3) *$G'(a_1) < 0$ and $G'(a_4) < 0$.*

These properties of $G(x)$ enable us to construct the following simple, closed curve $\Gamma(a)$. Define

$$\Phi_a^+(x) = \operatorname*{Max}_{x \leq \sigma \leq a} [G(\sigma)],$$
$$\Phi_a^-(x) = \operatorname*{Min}_{a \leq \sigma \leq x} [G(\sigma)]. \tag{6}$$

$\Gamma(a)$ is defined to be the simple, closed curve formed by $\Phi_{a_4}^+(x)$ and $\Phi_{a_1}^-(x)$ (see Fig. 5). Γ_2 of Fig. 3 was an example of such a curve. We use the letter "a" to refer to the

dependence of $\Gamma(a)$ upon the particular set $a = (a_1, a_2, a_3, a_4)$ satisfying II (1) and (2). The manner in which II (3) enters the discussion will be seen later.

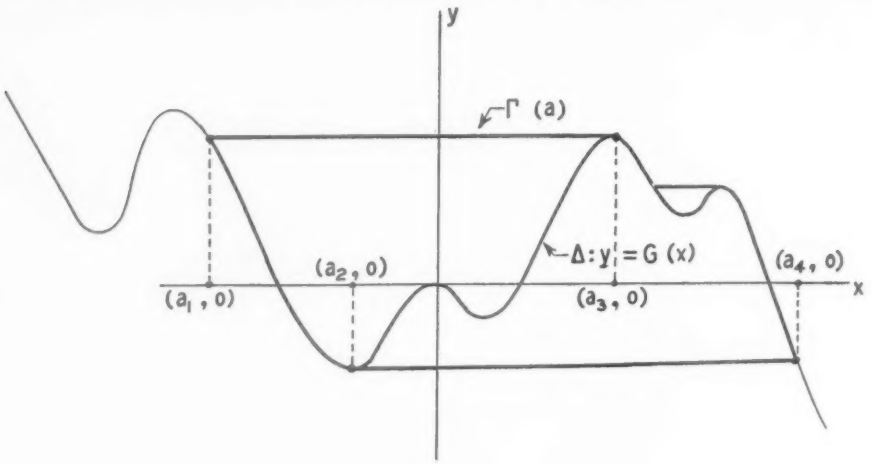

FIG. 5.

Let S and T be two regions of the phase plane. Define

$$d(S, T) = \underset{s \in S}{\text{l.u.b.}} \left\{ \underset{t \in T}{\text{g.l.b.}}\ d(s, t) \right\}^3$$

and

$$\delta(S, T) = d(S, T) + d(T, S).$$

By definition $S(\epsilon) \to T$ as $\epsilon \to 0$ if and only if $\delta(S(\epsilon), T) \to 0$ as $\epsilon \to 0$.

Theorem 1. *If $G(x)$ satisfies* I *and* II *and $0 < \epsilon < \epsilon_0$, an enclosure $E(a, \epsilon)$ containing a stable periodic solution of* (5) *can be constructed. As $\epsilon \to 0$, $E(a, \epsilon) \to \Gamma(a)$.*

Proof. Let us represent a simple closed curve C in the form

$$C: \quad \begin{aligned} x &= f(\theta), \\ y &= \psi(\theta) + g(x). \end{aligned} \tag{7}$$

Then

$$\bar{n} = \left(\frac{dy}{d\theta}, -\frac{dx}{d\theta} \right)$$

is a normal vector to C, and

$$\bar{v} = \left(\frac{dx}{dt}, \frac{dy}{d\theta} \right) = (G(x) - y, \epsilon^2 x)$$

is the phase velocity vector for a path γ through (x, y). Define $\lambda(\theta)$ to be the scalar product of $\bar{n}$ and $\bar{v}$, i.e.,

[3] $d(s, t)$ is the ordinary (Euclidean) distance between two points.

$$\lambda = [G(x) - \psi(\theta) - g(x)][\psi'(\theta) + g'(x)f'(\theta)] - \epsilon^2 f(\theta)f'(\theta). \tag{8}$$

The sign of λ is the sign of the projection of $\bar{v}$ on $\bar{n}$. The curves of interest to us are those for which $\lambda \geq 0$ or $\lambda \leq 0$. In the construction of the enclosure for this theorem we confine ourselves to the simpler solutions of the differential inequality.

Solution 1. Take $f(\theta) = \theta$ and $g(x) = 0$. Then

$$\lambda = [G(x) - \psi(x)]\psi'(x) - \epsilon^2 x.$$

(a) For $\psi(\theta) = A$, we have $D = -\epsilon^2 x$, and

$$C : y = A. \tag{9}$$

Hence $\lambda \leq 0$ for $x \geq 0$, and since the normal to C is vertical and downward, the paths for $x > 0$ cross horizontal lines from below. This, of course, we knew already. (See Fig. 3).

(b) Taking $\psi(x)\psi'(x) + \epsilon^2 x = A\psi'(x)$, we have

$$C : y = A \pm \epsilon(c^2 - x^2)^{1/2} \tag{10}$$

and

$$\lambda = \psi'(x)[G(x) - A] = \frac{\mp x}{(c^2 - x^2)^{1/2}}[G(x) - A].$$

The curve C is an ellipse with center at $(0, A)$ with c and ϵc as horizontal and vertical semi-axes. Along the lower-half of the ellipse the normal vector $\bar{n}$ is outward and on the upper-half, inward. Thus, for a particular x, we see from the expression for λ that the paths enter the ellipse if the horizontal axis of the ellipse is above Δ and leave the ellipse if the horizontal axis is below Δ.

Solution 2. Let $f(\theta) = \theta$ and $g(x) = \Phi_\alpha^+(x)$ (see (6)).

(a) $\psi(x) = 0$. Then

$$C : y = \Phi_\alpha^+(x), \qquad 0 \leq x \leq \alpha,$$

$$\lambda = -\epsilon^2 x$$

and $\lambda \leq 0$ for $x \geq 0$.

(b) $\psi(x) = \epsilon(\alpha^2 - x^2)^{1/2}$, i.e., $\psi(x)\psi'(x) + \epsilon^2 x = 0$. Here

$$\lambda = -\epsilon(\alpha^2 - x^2)^{1/2}\Phi_\alpha^{+\prime}(x) + \frac{\epsilon x}{(a^2 - x^2)^{1/2}}[\Phi_\alpha^+(x) - G(x)]$$

and from the definition of $\Phi_\alpha^+(x)$ it is clear that $\lambda \geq 0$ when $x \geq 0$. Hence the paths cross

$$C : y = \Phi_\alpha^-(x) + \epsilon(\alpha^2 - x^2)^{1/2}, \qquad 0 \leq x \leq \alpha, \tag{11}$$

from above.

The inner boundary (see Fig. 6). The inner boundary will be a simple, closed (sectionally smooth) curve $C_i(a, \epsilon)$ with the paths crossing $C_i(a, \epsilon)$ from its interior to exterior, the paths being tangent ($\lambda = 0$) at only a finite number of points.

(a) *Below* Δ *and* $x \geq 0$. (From A to B). Draw the line L_1 through A : $(0, G(a_2))$ with slope ϵ. The first intersection of L_1 with Δ is called B : $(b, G(b))$. It follows from II (2) that for ϵ sufficiently small the horizontal line from B to M lies below Δ and $b > a_3$. Take

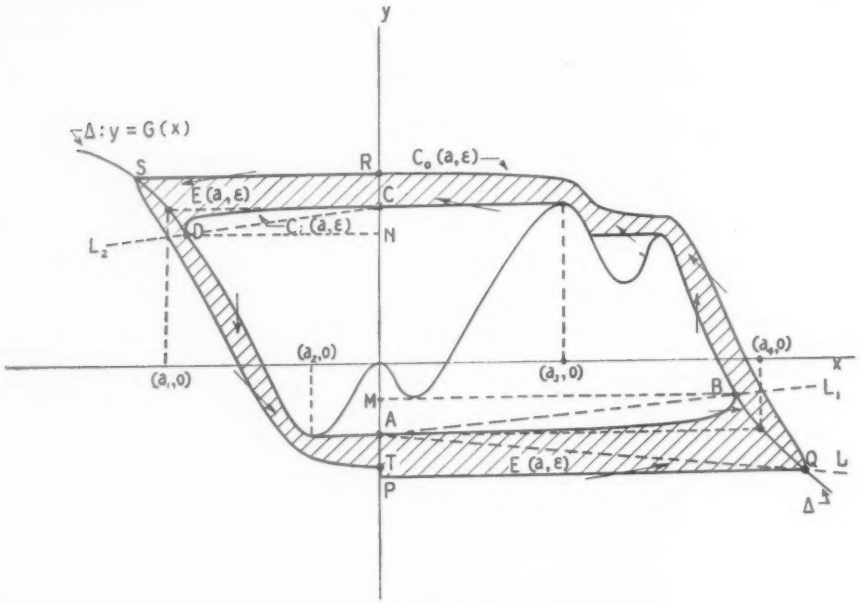

FIG. 6.

$$C_i(a, \epsilon) : y = G(b) - \epsilon(b^2 - x^2)^{1/2}, \qquad 0 \leq x \leq b.$$

This is an ellipse with center at M and semi-axes MA and MB.

(b) *Above or along* Δ *and* $x \geq 0$. (From B to C).

$$C_i(a, \epsilon) : y = \Phi_b^+(x), \qquad 0 \leq x \leq b.$$

(c) *Above* Δ *and* $x \leq 0$. (From C to D). Draw the line L_2 of slope ϵ through C : $(0, G(a_3))$. Its first intersection with Δ is D : $(d, G(d))$. For ϵ sufficiently small DN is above Δ and we take

$$C_i(a, \epsilon) : y = G(d) + \epsilon(d^2 - x^2)^{1/2}, \qquad d \leq x \leq 0.$$

(d) *Below or along* Δ *and* $x \leq 0$. (From D to A).

$$C_i(a, \epsilon) : y = \Phi_d^-(x), \qquad d \leq x \leq 0.$$

From solutions 1(b) and 2(a), and the remark made at the beginning of this section concerning the relation between the regions $x \geq 0$ and $x \leq 0$, the above is seen to be a suitable inner boundary.

The outer boundary: The outer boundary $C_0(a, \epsilon)$ will also be a simple closed (sectionally smooth) curve with the paths crossing $C_0(a, \epsilon)$ from its exterior to interior.

Again $\lambda = 0$ at only a finite number of points. Referring to Fig. 6 and II (3), we note first that within neighborhoods of $(a_1 , G(a_1))$ and $(a_4 , G(a_4))$ the inverse functions $G^{-1}(y)$ exist and have continuous derivatives. Consider the curves

$$C_1 : \xi = -G^{-1}(\eta + G(a_1))$$

and

$$C_4 : \xi = G^{-1}(-\eta + G(a_4)).$$

Figure 7 is the case $-a_1 = a_4$.) Draw the line L of slope $d\xi/d\eta = 1/\epsilon$. Since the slopes of C_1 and C_4 are bounded within a neighborhood of $\eta = 0$, it will be true for ϵ sufficiently small that, starting from 0 and going out along L, L will intersect both C_1 and C_4 and these intersections will be consecutive (or coincide). We assume that L intersects C_1 first and then C_4 or that they coincide, and begin the construction at A (Fig. 6). Otherwise we would start at C.

(a) *Below* Δ *and* $x \geq 0$. (From P to Q).

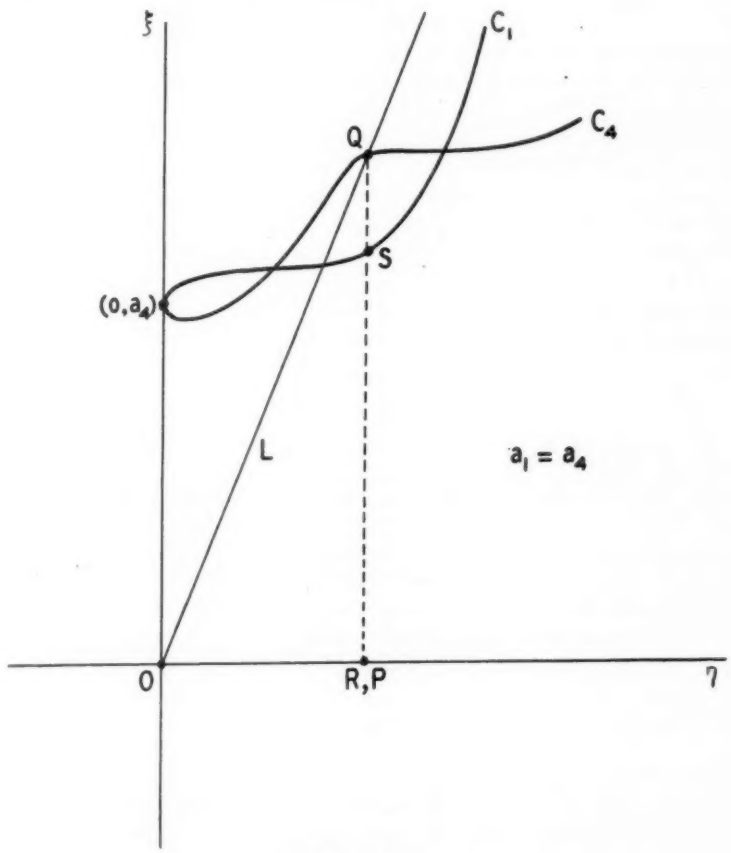

FIG. 7.*

*In Fig. 7, the relation between a_1 and a_4 should be $-a_1 = a_4$.

The line L of slope ϵ through A intersects Δ for ϵ sufficiently small. Call this first intersection $Q : (q, G(q))$.

$$C_0(a, \epsilon) : y = G(q), \qquad 0 \leq x \leq q$$

($q = \overline{PQ}$ of Fig. 6 or 7).

(b) *Above* Δ *and* $x \geq 0$. (From Q to R).

$$C_0(a, \epsilon) : y = G(q) + \epsilon(q^2 - x^2)^{1/2}, \qquad 0 \leq x \leq q.$$

(c) *Above* Δ *and* $x \leq 0$. (From R to S). The horizontal line from R to the left (Fig. 6) will intersect Δ for ϵ sufficiently small. Let this intersection be $S : (s, G(s))$. As noted above we have for ϵ sufficiently small that $-s = \overline{RS} < q = \overline{PQ}$. Take

$$C_0(a, \epsilon) : y = G(s) = G(a_3) + \epsilon q, \qquad s \leq x \leq 0.$$

(d) *Below* Δ *and* $x \leq 0$. (From S to T).

$$C_0(a, \epsilon) : y = \Phi_s^-(x) - \epsilon(s^2 - x^2)^{1/2}, \qquad s \leq x \leq 0.$$

T is the point $(0, G(a_2) + \epsilon s)$.

(e) *Along* $y = 0$. The curve is closed by the vertical line segment from T to P. Since $-s < q$, T is above P.

From solutions 1(a) and 2(b) and the manner in which the paths cross the y-axis, the above is a suitable outer boundary.

The only critical point of the system (5) is the origin (0, 0) and this point is not in $E(a, \epsilon)$. Therefore, as discussed previously, $E(a, \epsilon)$ contains a stable periodic solution and is an enclosure. Also it is clear that $E(a, \epsilon) \to \Gamma(a)$ as $\epsilon \to 0$. As a matter of fact $\delta(\Gamma(a), E(a, \epsilon)) = O(\epsilon)$.

For any given case the ϵ_0 of Th. 1 is the least upper bound of the ϵ meeting the conditions stated in the above construction. For van der Pol's equation ($G(x) = x - x^3/3$),

$$\epsilon_0 = \frac{2(3)^{1/2}}{9} < \frac{1}{2.6}.$$

Fig. 8 shows the enclosure for $\epsilon = 1/3$.

4. In van der Pol's special case it is possible to construct an enclosure for any $\epsilon > 0$ and thus to prove the existence of a stable periodic solution for each $\epsilon > 0$. Liénard [5] has, however, proved more; namely, that there exists for each $\epsilon > 0$ a unique periodic solution, and that this solution is stable. It is certainly not true in general that the periodic solution is unique. We might, however, expect it to be unique within $E(a, \epsilon)$. We now prove that this is the case.

Along a path γ of the system of equations

$$\begin{aligned} \frac{dx}{dt} &= G(x) - y \\ \frac{dy}{dt} &= \epsilon^2 x, \end{aligned} \tag{5}$$

$$\frac{1}{2}\frac{d}{dt}(\epsilon^2 x^2 + y^2) = \epsilon^2 x G(x). \tag{12}$$

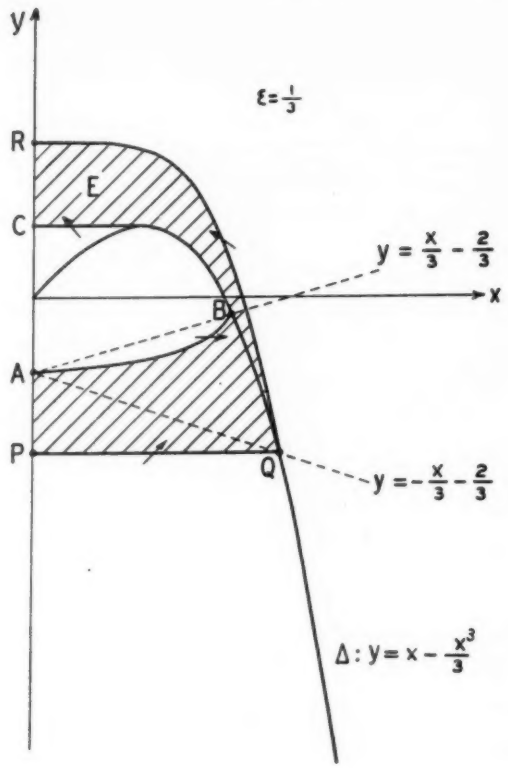

FIG. 8.

This equation expresses the relationship between the rate of change of the energy of the system and the power dissipation. Let $\gamma(b)$ be the path through the point $(0, b)$, $b < 0$, intersecting Δ and the y-axis as indicated in Fig. 9. Define

$$U(b) = \int_B^F \epsilon^2 x G(x)\, dt = \int_B^F G(x)\, dy \tag{13}$$

with the integration along $\gamma(b)$. Now by (12)

$$U(b) = \frac{1}{2}(OF^2 - OB^2) = \frac{1}{2}(f^2 - b^2) \tag{14}$$

and it is clear that a necessary and sufficient condition that $\gamma(b)$ be a periodic solution (a closed path) is that $U(b) = 0$—i.e., the change in energy over one cycle of the periodic solution is zero. For a given b_i , c_i and e_i are the x-coordinates of the intersection of $\gamma(b_i)$ with Δ.

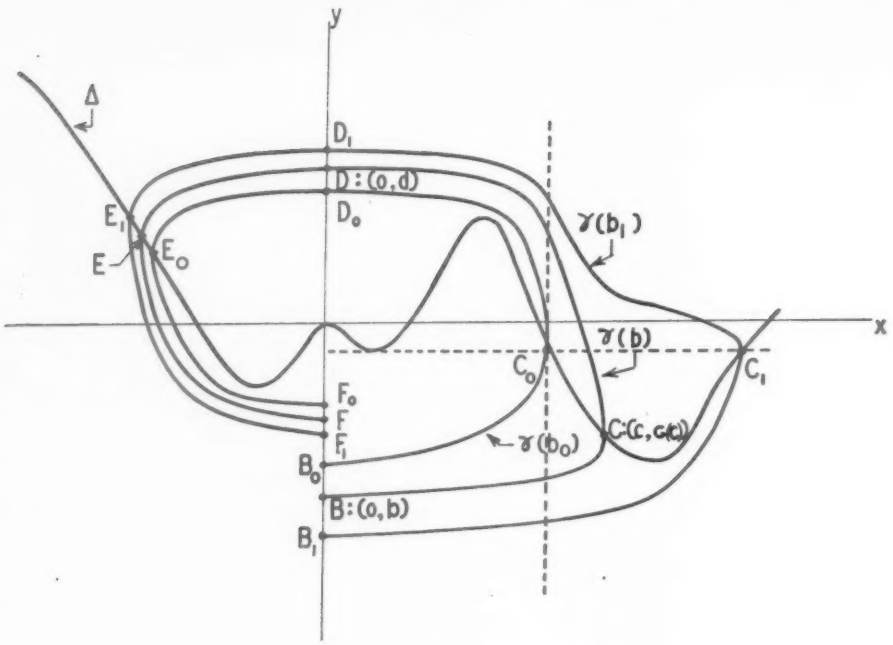

FIG. 9*

Theorem 2. *If* $G(x)$ *satisfies* I *of Sec.* 3 *and if, in addition,*

(a) $G(x) \geq G(c_0)$ *for* $0 \leq x \leq c_0$,

$G(x) \leq G(e_0)$ *for* $c_0 \leq x \leq 0$,

(b) $G(x) < G(c_0)$ *for* $c_0 < x < c_1$,

$G(x) > G(e_0)$ *for* $e_1 < x < e_0$;

then $U(b) < U(b_0)$ *for* $b_1 \leq b < b_0 < 0$.

Proof. Define (see Fig. 9)

$$U^+(b) = \int_B^D G(x)\, dy$$

and

$$U^-(b) = \int_D^F G(x)\, dy,$$

so that

$$U(b) = U^+(b) + U^-(b).$$

*The point F in Fig. 9 should be marked $F:(0, f)$.

Note also that

$$U^+(b) = \int_B^D [G(x) - G(c_0)]\, dy + \delta(b)G(c_0), \tag{15}$$

where $\delta(b) = d - b$ and the integral is along $\gamma(b)$. The paths $\gamma(b)$ do not intersect, and therefore $\delta(b)$ increases as b decreases. Let $\gamma_1(b)$ be that part of $\gamma(b)$ corresponding to $0 < x < c_0$ and $\gamma_2(b)$ that part of $\gamma(b)$ corresponding to $x > c_0$. On $\gamma_1(b)$, $G(x) - G(c_0) \geq 0$ and for the same value of x, we see from (5), since $c > c_0$, that

$$\frac{dy}{dx}\,(\text{on } \gamma_1(b)) < \frac{dy}{dx}\,(\text{on } \gamma(b_0)), \qquad \text{above } \Delta$$

and

$$\frac{-dy}{dx}\,(\text{on } \gamma_1(b)) < \frac{-dy}{dx}\,(\text{on } \gamma(b_0)), \qquad \text{below } \Delta.$$

Hence

$$\int_{\gamma_1(b)} [G(x) - G(c_0)]\, dy \leq \int_{\gamma(b_0)} [G(x) - G(c_0)]\, dy,$$

where the positive directions along the curves are those of increasing time (counter-clockwise). Thus from (14) and (15)

$$U^+(b) \leq U^+(b_0) + \int_{\gamma_2(b)} [G(x) - G(c_0)]\, dy$$

$$+ G(c_0)[\delta(b) - \delta(b_0)].$$

Now by (b), $G(x) - G(c_0) < 0$ on $\gamma_2(b)$; $G(c_0) \leq G(0) = 0$ by (a), and $\delta(b) - \delta(b_0) \geq 0$ since $b \leq b_0$. Hence

$$U^+(b) < U^+(b_0), \qquad b_1 \leq b < b_0 < 0.$$

In an exactly analogous fashion

$$U^-(b) < U^-(b_0), \qquad b_1 \leq b < b_0 < 0,$$

from which the conclusion of the theorem follows.

We can infer from this theorem that if $\gamma(b_0)$ were a periodic solution, then it would be the only periodic solution crossing the lower half of the y-axis between B_0 and B_1. A more important corollary of Th. 2 is

Theorem 3. *If* (a) $G(x)$ *satisfies I of Section* 3,

(b) $G(x) \geq G(c_2)$, $0 \leq x \leq c_2$,
$G(x) \leq G(e_2)$, $e_2 \leq x \leq 0$,

(c) $G(x)$ *is monotone decreasing for* $c_2 \leq x \leq c_3$ *and* $e_3 \leq x \leq e_2$;

then there is at most a single periodic solution $\gamma(b)$ *for* $b_3 \leq b \leq b_2 < 0$. *If the solution exists, it is stable.*[4]

Proof. If $b_3 \leq b_1 < b_0 \leq b_2$, then b_1 and b_0 satisfy the conditions of Th. 2 and $U(b_0) > U(b_1)$. Hence $U(b)$ is monotone increasing and has at most one zero for $b_3 \leq$

[4] A similar theorem holds for the unstable case and can be obtained by replacing $G(x)$ by $-G(x)$.

$b \leq b_2$; and there can be at most one periodic solution in the interval. From Eq. (14), it is clear that if there is a periodic solution it must be stable.

As an immediate consequence of this theorem, we can state that, *for each ϵ sufficiently small* ($0 < \epsilon < \epsilon_1$), *$E(a, \epsilon)$ of Theorem 1 contains a unique periodic solution and the periodic solution is stable.* The parameter ϵ_1 must be sufficiently small, $\epsilon_1 \leq \epsilon_0$, so that the parts of Δ contained in the interior of $E(a, \epsilon)$ have negative slope. By II such an ϵ_1 exists.

A singular case of some interest is that illustrated in Fig. 10. Here $\text{II}_{(3)}$ is not satisfied since the line AW is tangent to Δ at V. However, when $\epsilon < \epsilon_0$ the enclosure $E(a, \epsilon)$ can be constructed, although the inner and outer boundaries approach different closed curves. Thus we see that, for ϵ small, slight perturbations of the system would cause intermittent "jumps" from one mode of oscillation through V to another through W. If the motion of the representative point were photographed its trace would look some-

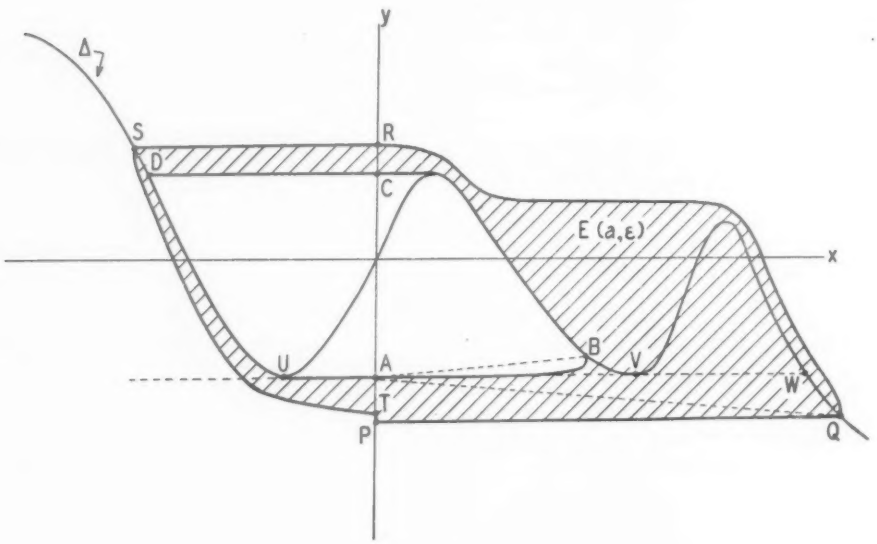

FIG. 10.

thing like that pictured in Fig. 11. This instability (thinking in terms of the LRC-circuit of Sec. 2) could be corrected by placing a linear resistance r in series with R. This would change the curve to a new curve $y = G(x) - rx$, would raise U, lower V and thus exclude this critical situation. We have then a simple mathematical example of "mode separation". In magnetrons mode separation is accomplished by the addition of rings or straps. Although this particular phenomenon is much more complex than our example and is not a relaxation oscillation, the example does illustrate the possibilities of this method of approach.

5. In addition to proving the existence of a periodic solution the foregoing construction of an enclosure provides certain quantitative information about the solution. For example, the knowledge that a periodic solution $\gamma(a, \epsilon)$ is in $E(a, \epsilon)$ immediately

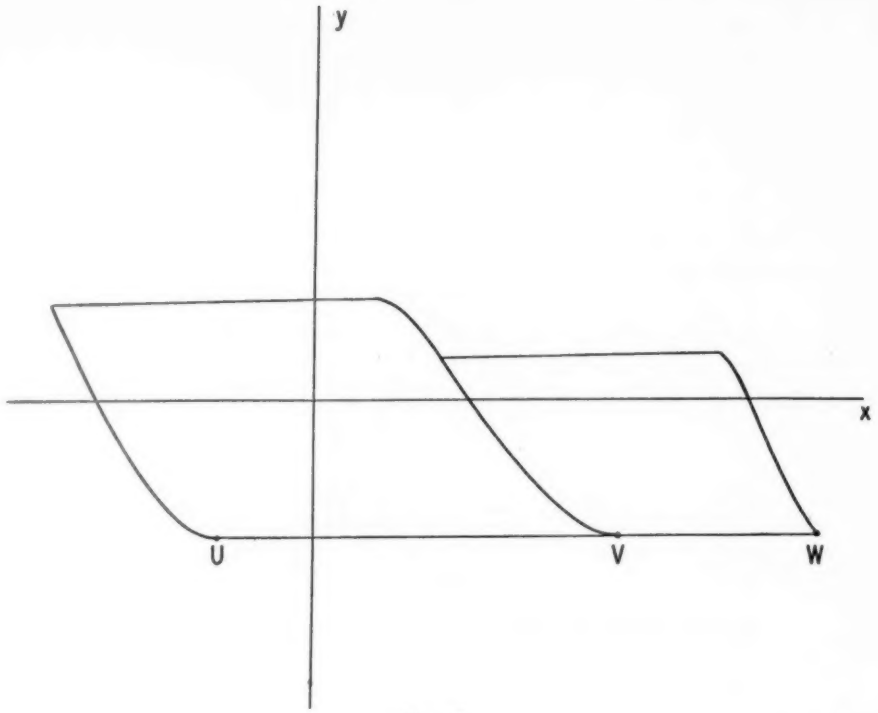

FIG. 11.

places bounds on its amplitude.[5] It is clear that these bounds are determined by the intersections (at Q, B, S, and D, Fig. 6) of $E(a, \epsilon)$ with Δ. With $G(x)$ given they could be readily computed. In van der Pol's case, $G(x) = x - x^3/3$ and

$$2 - \frac{2}{3}\epsilon - \frac{4}{27}\epsilon^2 \leq \text{amp} \leq 2 + \frac{2}{3}\epsilon - \frac{4}{27}\epsilon^2 \tag{16}$$

when $\epsilon \leq \epsilon_0 = 2(3)^{1/2}/9$. Hence $amp \to 2$ as $\epsilon \to 0$.

Of even greater importance is to investigate the asymptotic behavior of the period T of $\gamma(a, \epsilon)$. Van der Pol in [2] obtains graphically the result that $\epsilon^2 T$ is[6] approximately 2 for $\epsilon = 1/10$. Liénard in [5] gives an inconclusive proof that $\epsilon^2 T \to 3 - 2 \log 2 = 1.6 +$ and Kazakevich [12] gives the result as $\epsilon^2 T \to 2$. As we shall see, Liénard's result is correct.[7]

Let $\gamma(b, \epsilon)$ be, as in Fig. 12, the path through the point $B : (0, b)$, $b < 0$.[8] In order to assure that for all $\epsilon > 0$, $\gamma(b, \epsilon)$ intersects Δ we assume that for some $x = c_3 > 0$, $G(c_3) = b$. Let T_1 be the time for the representative point to move from B to the point C on Δ. L_1 is the line through B with slope ϵ. C_1 is the first intersection ($x > 0$) of L_1

[5] The amplitude is the maximum of $|x|$.

[6] Our units differ from van der Pol's by the factor $1/\epsilon$.

[7] A refinement of this result is given by Haag [7].

[8] The case $b > 0$ can be treated in a similar manner, as was pointed out at the beginning of Sec. 3.

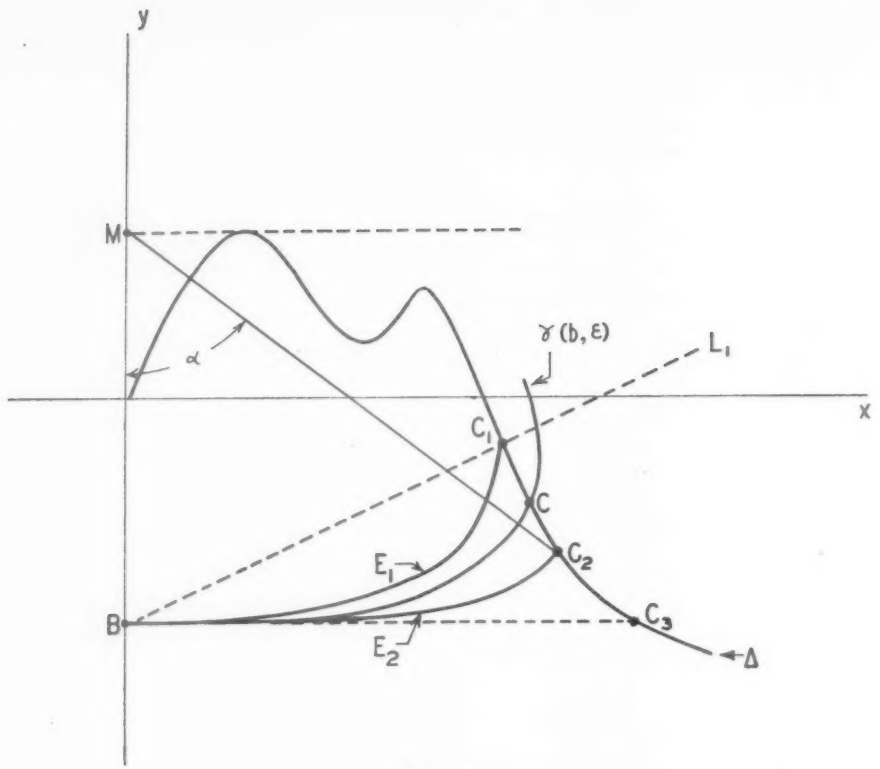

FIG. 12.

with Δ, and D: $(d, G(d))$ is an absolute maximum of $G(x)$ for $0 \leq x \leq c$. Then by solution (1) of Th. 1 $\gamma(a, \epsilon)$ lies between the ellipses[9]

$$E_1: \quad \begin{aligned} x &= c_1 \cos\theta \\ y &= G(c_1) + \epsilon c_1 \sin\theta, \qquad \frac{-\pi}{2} \leq \theta \leq 0 \end{aligned}$$

and

$$E_2: \quad \begin{aligned} x &= \frac{d-b}{\epsilon} \cos\varphi \\ y &= d + (d-b)\sin\varphi, \qquad -\frac{\pi}{2} \leq \varphi \leq -\frac{\pi}{2} + \varphi_2 \end{aligned}$$

where $\varphi = -\pi/2 + \varphi_2$ corresponds to the intersection C_2 of E_2 with Δ. Thus $\varphi_2 = \tan^{-1}(\epsilon \tan\alpha)$, where α is the angle BMC_2. From Eq. (5)

$$dt = \frac{dy}{\epsilon^2 x}$$

[9]We are assuming ϵ to be so small that the horizontal axis of E_1 lies below Δ.

and hence

$$\int_{E_2} \frac{dy}{x} \leq \epsilon^2 T_1 = \int_{\gamma(b)} \frac{dy}{x} \leq \int_{E_1} \frac{dy}{x}.$$

It then follows that

$$\epsilon\varphi_2 \leq \epsilon^2 T_1 \leq \epsilon \frac{\pi}{2}. \tag{17}$$

In order to treat the case where the path is above Δ we need the following solution to $\lambda \geq 0$.

Solution 3. Taking $f(\theta) = \theta$, $g(x) = \Phi_c^+(x)$ and $\psi = k_c(c^2 - x^2)^{1/2}$ in Eq. (7), we obtain

$$\Gamma : y = \Phi_c^+(x) + \epsilon k_c(c^2 - x^2)^{1/2}, \qquad 0 \leq x \leq c \tag{18}$$

and

$$\lambda = x\left\{\epsilon^2 k_c^2 + \left[\frac{\Phi_c^+(x) - G(x)}{(c^2 - x^2)^{1/2}} - \frac{\Phi_c^{+\prime}(x)(c^2 - x^2)^{1/2}}{x}\right]\epsilon k_c - \epsilon^2\right\}.$$

Noting that

$$\lim_{x \to c} - \frac{\Phi_c^+(x) - G(x)}{(c^2 - x^2)^{1/2}} = 0,$$

we define

$$2M_c = \operatorname*{Max}_{0 \leq x \leq c}\left[\frac{\Phi_c^+(x) - G(x)}{(c^2 - x^2)^{1/2}} - \frac{\Phi_c^{+\prime}(x)(c^2 - x^2)^{1/2}}{x}\right] < \infty$$

and obtain

$$\lambda \leq x(\epsilon^2 k_c^2 + 2M_c \epsilon k_c - \epsilon^2).$$

Picking

$$k_c = -\frac{M_c}{\epsilon} + \left[\left(\frac{M_c}{\epsilon}\right)^2 + 1\right]^{1/2} = \frac{\epsilon}{2M_c} - \left(\frac{\epsilon}{2M_c}\right)^3 + O(\epsilon^5),$$

we find that $\lambda \leq 0$ along Γ.

Therefore from the above and the solution (2) of Th. 1, the curve $\gamma(c, \epsilon)$ (Fig. 13) from $C : (c, G(c))$ to the y-axis at H lies between Γ of (18) and

$$\Gamma_1 : y = \Phi_c^+(x) + \epsilon(c^2 - x^2)^{1/2}. \tag{19}$$

Let D be the absolute maximum point of Δ, $0 \leq x \leq c$, which is closest to the y-axis; and E the intersection with $\gamma(c, \epsilon)$ of the horizontal line through D.

Now by (5)

$$\epsilon^2 T_2 = \int_{C_\gamma(c,\epsilon)}^{E} \frac{dy}{x} + \int_{E_\gamma(c,\epsilon)}^{F} \frac{dy}{x} + \epsilon^2 \int_{F_\gamma(c,\epsilon)}^{H} \frac{dx}{G(x) - y},$$

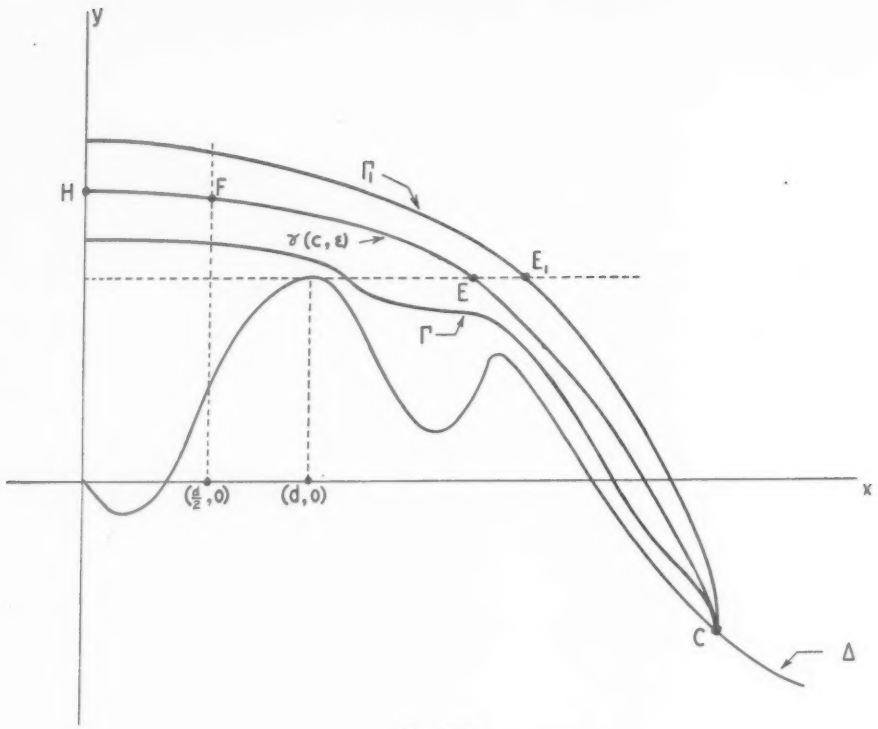

FIG. 13.

where F corresponds to the point $x = d/2$ on $\gamma(c, \epsilon)$. But

$$\int_{C\gamma(c,\epsilon)}^{E} \frac{dy}{x} \leq \int_{C\Gamma}^{D} \frac{dy}{x} = \int_{c}^{d} \frac{\Phi_c^{+\prime}(x)}{x}\,dx + k_c\epsilon \int_{c}^{d} \frac{dx}{(c^2 - x^2)^{1/2}}$$

$$\int_{E\gamma(c,\epsilon)}^{F} \frac{dy}{x} \leq \int_{G(d)}^{G(d)+\epsilon(c^2-d^2/4)^{1/2}} 2\,\frac{dy}{d} = \frac{2\epsilon}{d}\,(c^2 - d^2/4)^{1/2}$$

$$\int_{F\gamma(c,\epsilon)}^{H} \frac{dx}{G(x) - y} \leq \int_{0}^{d/2} \frac{dx}{G(d) - G(x) + k_c\epsilon(c^2 - x^2)^{1/2}}$$

$$\leq \int_{0}^{d/2} \frac{dx}{G(d) - G(x)}\,,$$

and hence

$$\epsilon^2 T_2 \leq \int_{c}^{d} \frac{\Phi_c^{+\prime}(x)}{x}\,dx + O(\epsilon). \tag{20}$$

Since $\gamma(c, \epsilon)$ lies below Γ_1 of (19),

$$\epsilon^2 T_2 = \int_{\gamma(c,\epsilon)} \frac{dy}{x} \geq \int_{C\Gamma_1}^{E_1} \frac{dy}{x} = \int_{c}^{e_1} \frac{\Phi_c^{+\prime}(x)}{x}\,dx + \epsilon \cos^{-1}\frac{e_1}{c} \tag{21}$$

where E_1 is the point on Γ_1 for which $y = G(d)$ and e_1 is the x-coordinate of E_1. Hence

$$-\int_{e_1}^{c} \frac{\Phi_c^{+\prime}(x)}{x}\,dx + O(\epsilon) \leq \epsilon^2 T_2 \leq -\int_{d}^{c} \frac{\Phi_c^{+\prime}(x)}{x}\,dx + O(\epsilon). \tag{22}$$

Note that $e_1 \to d$ as $\epsilon \to 0$. Bounds on the times when $x < 0$ can be obtained in a similar manner. Thus we see from (17) and (22) that the time $\epsilon^2 T$ which the path spends in the region above where $G'(x) < 0$ for $x > 0$ and below where $G'(x) < 0$[10] for $x < 0$ is $O(1)$. Otherwise the time is $O(\epsilon)$. Thus for small ϵ we obtain the characteristic of a relaxation, with x changing relatively slowly in some regions and rapidly elsewhere.

Let $T(a, \epsilon)$ be the period of the periodic solution within $E(a, \epsilon)$. Referring to (22) and Fig. 6, we see that

$$\epsilon^2 T(a, \epsilon) \to \int_{a_1}^{a_4} \frac{\Phi_{a_1}^{-\prime}(x) - \Phi_{a_4}^{+\prime}(x)}{x}\,dx \qquad \text{as} \qquad \epsilon \to 0. \tag{23}$$

In van der Pol's case ($G(x) = x - x^3/3$) we obtain

$$\epsilon^2 T \to 2\int_1^2 \left(x - \frac{1}{x}\right) dx = 3 - 2\log 2 = 1.614+,$$

which confirms the result given by Liénard.

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[10] $G'(x) < 0$ states that the dynamic resistance is positive.

SURFACE WAVES*

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Introduction. The theory of low gravity waves deals with the oscillations of a fluid under constant pressure when the inertia, viscous and capillary forces of the motion are small compared with gravitational forces. The study derives its interest particularly from problems of the propagation of waves, oscillations of ships in waves, and the design of harbors and breakwaters. The characteristics of gravity waves mentioned above are more or less satisfied in these problems, at any rate, if we consider regions sufficiently far from the breaker zone and the storm area where the waves are generated.

In the present paper we are concerned with two dimensional motion, that is, a motion where the crests of the waves are straight and parallel. Such motion may be expected to occur in regions whose distance from the storm area is large compared with the diameter of the storm area and with the diameter of the region considered.

In the text books, e.g. H. Lamb, *Hydrodynamics*, only wave motions in a canal of uniform depth are studied, and it is further supposed that the pressure is constant over the whole upper surface of the fluid. Now, in fact, not only do waves travel over uneven bottoms, but in some of the problems mentioned above we are interested just in what happens when there are obstacles in the bottom or on the surface, e.g. in the

FIG. 1.

reflection[1] set up by reefs, moored ships, or breakwaters. Recently, reflection of waves has been studied by Dean, Lewy, Stoker, Ursell and others[2]. The work concentrated on motion in water very deep or very shallow compared with the wave length. These conditions are appropriate for studying ships in waves or the behavior of waves very near the coast. But while such investigations may give important information on how the reflection depends on the cross sectional shape of the obstacles (in deep or shallow water), naturally they say nothing about the effect of depth, and such information is

*Received Jan. 30, 1948.

[1]Another practically important problem is to determine the effect of the waves on the obstacle, the mean pressures set up by the waves. This does not seem to come out of the linear theory or simple modifications of it.

[2]Cf. e.g. J. J. Stoker, *Surface waves in water of variable depth*, Quart. Appl. Math. **5**, 1-54, 1947, F. Ursell, *The effect of a fixed vertical barrier on surface waves in deep water*, Proc. Camb. Phil. Soc. **43**, 3 1947; and references given there

necessary to decide, for example, where breakwaters are most effective (for waves of given period). We therefore try to marshal the reflection of obstacles in *arbitrary* depth, that is, we want to determine the reflected and transmitted waves system for a wave train coming up against (cylindrical) obstacles which lie in the bottom of a canal or are fixed in the surface.

Throughout the paper we consider simply harmonic oscillations; for, from such work the reflection of arbitrary incoming motion can be calculated, provided the reflection of incoming waves of short wave lengths is sufficiently small;[3] and if the reflection varies *slowly* with the wave length, the reflected wave at a point P at one time is determined by the incoming motion at P during a *short* interval. Among our results there will be only one where the reflection is sensitive to the wave length, that is, it is not well-defined by the incoming waves (if viscous and inertia forces are neglected).

Also we suppose that the depth of the canal is constant sufficiently far to either side of the obstacles.

Summary. In Sec. 1 we consider the *asymptotic behavior of wave motions and the definition of the reflection coefficient.* It was implicitly assumed above (where we spoke of incoming and reflected wave systems) that the wave motion is asymptotically a superposition of simple wave trains. More precisely we assume: if $\phi e^{i\sigma t}$ is the potential of a wave motion of period $2\pi/\sigma$ defined in the domain of the fluid, whose normal derivative vanishes on the bottom and on the cylinders, and whose pressure is nearly constant on the free surface ($\sigma^2\phi - g\,\partial\phi/\partial y = 0$ on the mean free surface, where y is measured along the vertical), then ϕ is asymptotically of the form $(ae^{ikx} + be^{-ikx})\cosh k(y + h)$ (at the right hand infinity, say) where $\sigma^2 = gk \tanh kh$, and h is the right hand asymptotic depth of water. Further, in asking for *the* reflection we assume that a unique transmitted and reflected wave is consistent with a prescribed incoming wave. These assumptions are correct if and only if ϕ is restricted to be bounded.

We show in Th. I, using an expansion theorem of A. Weinstein, that at a few depths from the obstacle the potential is very nearly a superposition of simple wave trains, the error falling off exponentially with distance. Further, if for a given domain of fluid there is a potential which is not asymptotically zero, a right hand and a left hand reflection coefficient can be defined; they are unique, and equal to one another. (We calculate generally the left hand reflection coefficient.)

It can be shown that these results hold even if the depth is not constant, but only nearly so (differs exponentially little from a constant).

Section 2 provides a *general reduction of the problem.* To calculate the reflection coefficient for given obstacles a mixed boundary value problem for a potential in the domain of the fluid has to be solved and its asymptotic form determined. Since the solution of linear boundary value problems in a rectangular strip is relatively simple we transform the domain of the fluid into a strip whose width is equal to the asymptotic depth which from now on is assumed to be the same at either infinity. In Lemma I the problem is reduced to a linear integral equation for the potential on one boundary of the strip, and in Lemma II a solution by iteration is given (which converges for suitable obstacles).

In Sec. 3 we apply the results of Sec. 2 to determine the reflection from *obstacles in the bottom* where the surface remains free. We show in Th. I that if $z(\zeta)$ is a conformal

[3]For a discussion of the conditions see a forthcoming paper, *Some remarks on integral equations with kernels* $L(x_1 - \xi_1, \ldots, x_n - \xi_n)$, particularly Sec. 4.

transformation of the strip $0 > \eta > -h$ into the domain of the fluid where the infinities correspond and

$$\alpha = \frac{2k}{1 + 2kh/\sinh 2kh} \int_{-\infty}^{\infty} \left| \frac{dz}{d\zeta} - 1 \right|_{\eta=0} d\xi + \max_{\eta=0} \left| \frac{dz}{d\zeta} - 1 \right| \frac{1 - 2kh/\sinh 2kh}{1 + 2kh/\sinh 2kh} < 1,$$

ϕ can be computed by iteration, and the reflection coefficient R is given by

$$\frac{R_0 - \alpha^2/2(1-\alpha)}{\{1 + [R_0 - \alpha^2/2(1-\alpha)]^2\}^{1/2}} < R < \frac{R_0 + \alpha^2/2(1-\alpha)}{\{1 + [R_0 + \alpha^2/2(1-\alpha)]^2\}^{1/2}}$$

where

$$R_0 = \frac{k}{1 + 2kh/\sinh 2kh} \left| \int_{-\infty}^{\infty} \left(\frac{dz}{d\zeta} - 1 \right)_{\eta=0} \exp(2ik\zeta)\, d\zeta \right|.$$

In general, the α of the obstacle in which we are interested will not be easily calculated. It is therefore desirable to show how α varies with the shape of the obstacle. This is done in two (simple) general theorems.

Theorem II: If the domains D_1 , D_2 both lie in the strip $0 > y > -h$, and D_1 is included in D_2 , we have

$$\alpha_1 \geq \alpha_2 .$$

FIG. 2.

Theorem III: If D_1 , D_2 are the intersection and join of a striplike domain D with the strip $0 > y > -h$, we have

$$\alpha \leq \alpha_1 + \alpha_2 .$$

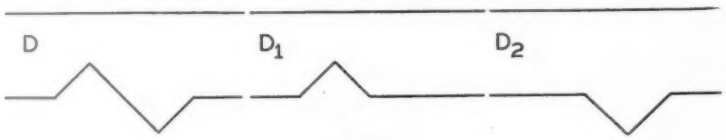

FIG. 3.

In Sec. 4 *obstacles in the surface* are considered. If the draught of an obstacle in the surface is small compared with the depth and the wave length, and the beam is moderately short compared with the wave length, the reflection from the obstacle tends to the reflection by a "plate" of the same beam in the surface, and

$$\frac{2k[(\sin 2k)/2k - \alpha^2/(1-\alpha)]/(1 + 2kh/\sinh 2kh)}{\{1 + 4k^2[(\sin 2k)/2k - \alpha^2/(1-\alpha)]^2/(1 + 2kh/\sinh 2kh)^2\}^{1/2}}$$

$$< R < \frac{2k[(\sin 2k)/2k + \alpha^2/(1-\alpha)]/(1 + 2kh/\sinh 2kh)}{\{1 + 4k^2[(\sin 2k)/2k + \alpha^2/(1-\alpha)]^2/(1 + 2kh/\sinh 2kh)^2\}^{1/2}}$$

where

$$\alpha < 4 \sin^2 k/(1 + 2kh/\sinh 2kh) + (1 - 2kh/\sinh 2kh)/(1 + 2kh/\sinh 2kh) < 1$$

or

$$\alpha < 4 \sin^2 k/(1 + 2kh/\sinh 2kh) + 2[2^{-1/2} \exp(-2^{1/2}kh) + 1/kh + 2\pi^{-1/2}k^{1/2}] < 1$$

and $k = \pi$ *beam/wave length*. The former bound for α is useful in shallow water, the latter in deep water.

Discussion. Throughout, explicit bounds for the error in our calculations are given, and the iterative method is expected to be of use in numerical work, particularly in the problem of Sec. 4 where we get an integral equation over a finite range. The bounds are good in shallow water since then the term $(1 - 2kh/\sinh 2kh)/(1 + 2kh/\sinh 2kh)$ in α is small. But it seems worthwhile to collect some results which are easily obtained from the algebra without further calculations.

The reflection coefficient for obstacles in the bottom takes a particularly simple form if the wave length is large compared with the depth and with the dimensions of the obstacle. Then

$$R_0 = \frac{k}{1 + 2kh/\sinh 2kh} \left| \int_{-\infty}^{\infty} \left(\frac{dz}{d\zeta} - 1\right)_{\eta=0} \exp(2ik\zeta)\, d\zeta \right|,$$

and if the waves are long, $2k\zeta$ is nearly constant over the range where $(dz/d\zeta - 1)_{\eta=0}$ is noticeable, so that

$$R_0 \sim \tfrac{1}{2}k \left| \int_{-\infty}^{\infty} \left(\frac{dz}{d\zeta} - 1\right) d\zeta \right|,$$

i.e., the reflection coefficient is $k/2$ times the contraction constant of the transformation which maps the strip $0 > \eta > -h$ into the domain of the fluid.

We compare the reflection by a vertical barrier and a long low reef.

In Sec. 3, we determine the transformation function $z(\zeta)$ for a strip of width h with a cut of length r in the bottom (a barrier of height r). Then α becomes

$$\frac{8kh}{\pi(1 + 2kh/\sinh 2kh)} \log \sec \frac{r\pi}{2h} + 2 \sin^2 \frac{r\pi}{4h} \frac{1 - 2kh/\sinh 2kh}{1 + 2kh/\sinh 2kh},$$

and if $kh \to 0$ we have

$$R_0 \to \frac{2}{\pi} kh \log \sec \frac{r\pi}{2h}$$

To get the reflection by a low gently sloping reef of the form $y = -h + \epsilon f(x)$ we use an approximate expression for $z(\zeta)$ in terms of $f(x)$ to show that

$$R_0 \sim \frac{\epsilon}{h} \frac{2kh}{\sinh 2kh(1 + 2kh/\sinh 2kh)} k \left| \int_{-\infty}^{\infty} f(x) \exp(2ikx)\, dx \right|.$$

For a horizontal reef of width a and height ϵ

$$R_0 \sim \frac{\epsilon}{h} \frac{2kh\, |\sin 2ka|}{\sinh 2kh(1 + 2kh/\sinh 2kh)}.$$

(No details are given in the paper.)

If the width of the reef is large compared with the wave length, R_0 varies rapidly with the wave length. To calculate the reflection at a point P at one time we have to know the incoming motion at P over an interval of order $a(gh)^{-1/2}$, for fixed ϵ/h.

In practical problems it is often important to understand how the reflection of a *given* obstacle in the incoming waves varies when it is placed in different depths. The period of the waves remains constant so that it is necessary to plot the reflection against the depth. Since however kh is a monotone function of h for any σ^2, we get the general shape of the curve by plotting the reflection against kh.

We observe that the reflection by a horizontal reef of width a decreases rapidly as the depth increases, provided $ka < \pi/2$.

$$\left(k = \frac{\sigma^2}{g \tanh kh} \quad \text{and} \quad R_0 \sim \frac{\epsilon\sigma^2}{g} \frac{\sin(2\sigma^2 a/g \tanh kh)}{\sinh^2 kh(1 + 2kh/\sinh 2kh)}\right).$$

Lastly consider the variation of the reflection of waves of given period by a plate of given beam in the surface of water,

$$R \sim \frac{k \times \text{beam}}{1 + 2kh/\sinh 2kh'} \quad \text{i.e.,} \quad R \sim \frac{\sigma^2 \times \text{beam}}{g(\tanh kh + kh \operatorname{sech}^2 kh)}.$$

We find that the reflection first decreases and then increases as we go from shallow into deep water (or vice versa):

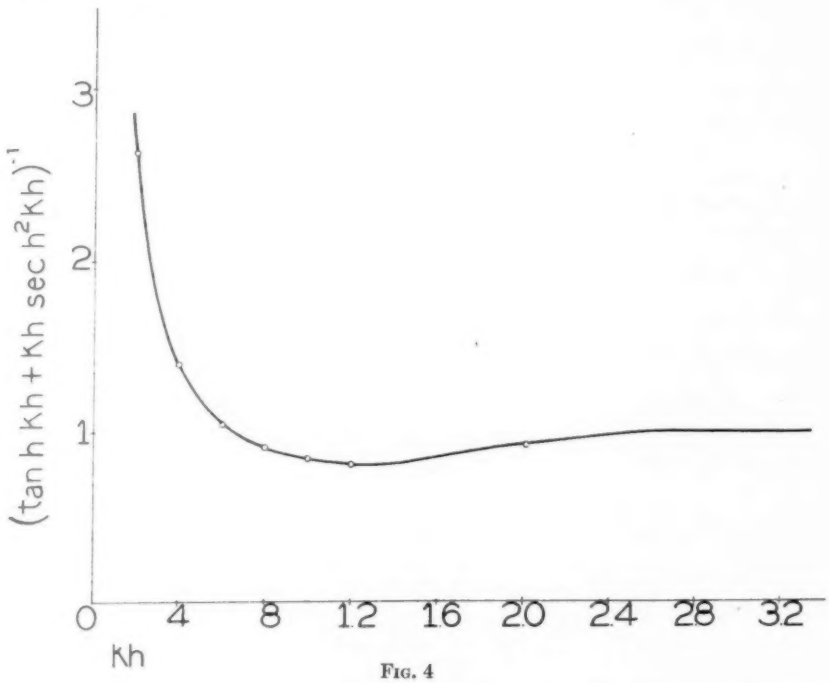

FIG. 4

But more tedious algebra would be needed to give information on the effect of draught in different depths of water (Th. II of Sec. 4). Also, what happens when the

dimensions of the obstacle are nearly equal to the wave length is undecided by the present work.

1. Asymptotic behavior and definition of the reflection coefficient. To discuss two dimensional wave motion in a domain sketched in Fig. 1[4] we choose a system of coordinates so that the z-axis is parallel to the axes of the cylinders, the mean free surface lies in $y = 0$, and y is measured vertically upwards.

By the theory of small oscillations the potential $\phi(x, y)e^{i\sigma t}$ must satisfy the following conditions:

(A) ϕ is bounded and harmonic in the domain of the fluid,
$\partial\phi/\partial n = 0$ on the lower boundary and on the fixed cylinders, where $\partial/\partial n$ is the derivative along the normal to the boundary,
$\partial\phi/\partial y$ is bounded in the neighborhood of the mean free surface,
$\sigma^2\phi - g\,\partial\phi/\partial y = 0$ on the mean free surface, where g denotes the acceleration due to gravity.

In analyzing possible motions satisfying (A) it is convenient to use the following lemmas:

Lemma I (A. Weinstein, C. R., 1927). The functions

$$\{\cosh k(y + h), \cos k_n(y + h)\}$$

are a complete set of orthogonal functions which satisfy

$$\sigma^2 f_n - g\, df_n/dy = 0 \quad \text{for } y = 0,$$

$$df_n/dy = 0 \quad \text{for } y = -h,$$

$$f_n \in L(-h, 0)$$

if $\sigma^2 = gk \tanh kh$, $\sigma^2 = -gk_n \tan k_n h$, $(n - 1/2)\pi < k_n h < n\pi$.

Lemma II (ibid.). A harmonic function $\phi(x, y)$, bounded in the strip $0 > y > -h$, which satisfies

$$\sigma^2\phi - g\,\partial\phi/\partial y = 0 \text{ for } y = 0$$

$\partial\phi/\partial y$ bounded in the strip and zero on $y = -h$,

is of the form

$$(ae^{ikx} + be^{-ikx}) \cosh k(y + h).$$

Lemma III. A harmonic function $\phi(x, y)$, bounded in the half strip $0 > y > -h$, $x < X_0$, which satisfies

$\partial\phi/\partial y$ bounded in the half strip and zero on $y = -h$, $r < X_0$,
$\sigma^2\phi - g\,\partial\phi/\partial y$ prescribed on $y = 0$, $x < X_0$,
ϕ prescribed on $x = X_0$, $0 > y > -h$,

is defined uniquely except for a term

$$a \sin k(x - X_0) \cosh k(y + h).$$

[4] The (striplike) domains considered below are (1) bounded by $y = 0$ for $|x| > X_0$; $y = -h$ for $x < -X_0$; $y = -h'$ for $x > X_0$; a Jordan arc joining $(\pm X_0, 0)$; a Jordan arc joining $(-X_0, -h)$ to $(X_0, -h')$; (2) the boundary is a simple curve.

For, the difference Φ between two expressions satisfying these conditions is bounded in the half strip; also

$$\begin{aligned} &\partial\Phi/\partial y && \text{is bounded and zero on } y = -h,\ x < X_0\ , \\ &\sigma^2\Phi - g\,\partial\Phi/\partial y = 0 && \text{on } y = 0,\ x < X_0\ , \\ &\Phi = 0 \text{ on} && x = X_0\ ,\ 0 > y > -h. \end{aligned}$$

Since Φ is bounded[5] near the line $x = X_0\ ,\ 0 > y > -h$, and zero on it, Φ can be continued across it, and $\Phi(X_0 + x,\ y) = -\Phi(X_0 - x,\ y)$. Thus Φ satisfies the conditions of Lemma II in the whole strip $0 > y > -h$, and is zero on $x = X_0$. Hence the lemma. The result also holds if $X_0 = \infty$.

Theorem I. If a potential satisfies (A), then over the flat portion at either infinity it is of the form

$$(ae^{ikx} + be^{-ikx}) \cosh k(y + h) + \sum_1^\infty a_n \exp(-k_n \mid x \mid) \cos k_n(y + h), \tag{1.1}$$

where h is the asymptotic depth at the infinity considered. Also

$$\sum_1^\infty \mid a_n \mid \exp(-k_n \mid x \mid) = O(\exp\{-\pi \mid x \mid /2h\}).$$

Consider the flat portion at the left hand infinity.

Along any vertical $x = -X_0\ ,\ 0 > y > -h$, by Lemma I, ϕ can be expanded in the form

$$\phi(-X_0\ ,\ y) = \alpha \cosh k(y + h) + \sum_1^\infty \alpha_n \cos k_n(y + h), \tag{1.2}$$

$$h(1 + \sinh k_n h/2k_n h)\alpha_n = \int_{-h}^{0} \phi(-X_0\ ,\ y) \cos k_n(y + h)\, dy. \tag{1.3}$$

By analytic continuation across $y = 0$ and $y = -h$, ϕ is analytic on $x = -X_0\ ,\ 0 \geq y \geq -h$, so that repeated integration by parts of

$$\int_{-h}^{0} \phi(-X_0\ ,\ y) \cos k_n(y + h)\, dy$$

is allowed, and $\alpha_n = O(n^{-4})$. Thus (1.2) is uniformly convergent, and the series

$$(a_0e^{ikx} + b_0e^{-ikx}) \cosh k(y + h) + \sum_1^\infty \alpha_n e^{k_n\pi} e^{k_n X_0} \cos k_n(y + h), \tag{1.4}$$

with

$$a_0e^{ikX_0} + b_0e^{-ikX_0} = \alpha$$

is bounded and harmonic in $x < -X_0\ ,\ 0 > y > -h$, and equal to (1.2) on $x = -X_0$. By Lemma III, (1.4) is the potential in the half strip $0 \geq y \geq -h,\ x < -X_0$. Similarly for the right hand infinity.

Since $(n - 1/2)\pi < k_n h < n\pi$, the infinite series is less than

$$\exp(\pi \mid x \mid /2h) \sum_1^\infty \mid \alpha_n \mid \exp(-\pi \mid x + X_0 \mid /h).$$

[5] The symmetry principle for harmonic functions does not require the function to be continuous near the arc across which it is continued, but it is sufficient that it is bounded.

If $|\phi(-X_0, y)|_{0>y>-h} < M$, $|\alpha_n| < 2M/(1 - 1/\pi)$ by (1.3), and the series is less than $2M(1 - 1/\pi)^{-1} \exp[-1/2\,\pi(|x| - X_0)/h]\{1 - \exp[-\pi(|x| - X_0)/h]\}^{-1}$.

Theorem II. Suppose at the two infinities the asymptotic forms of a potential satisfying (A) are

$$(ae^{ikx} + be^{-ikx}) \cosh k(y + h), \qquad (a'e^{ik'x} + b'e^{-ik'x}) \cosh k'(y + h').$$

Then

$$(|a|^2 - |b|^2)kh(1 + \sinh 2kh/2kh) = (|a'|^2 - |b'|^2)k'h'(1 + \sinh 2k'h'/2k'h').$$

This is the principle of the constancy of transmission of energy.

Let X_0 be so large that the bottom is flat for $|x| \geq X_0$. If $\bar{\phi}$ is the complex conjugate of ϕ, $\bar{\phi}$ also satisfies (A). Consider

$$\int_C (\phi \nabla \bar{\phi} - \bar{\phi} \nabla \phi) \cdot d\mathbf{S},$$

where C is the contour of the domain of fluid between $x = \pm X_0$, and $d\mathbf{S}$ is a surface element of the boundary. Since ϕ and $\bar{\phi}$ are harmonic and bounded in the interior of C, the integral is zero by Green's theorem. The integrand is zero on the lower boundary and on the fixed cylinders since $\nabla\phi \cdot d\mathbf{S} = 0$, and on the mean free surface since

$$\nabla\phi \cdot d\mathbf{S} = \frac{\partial \phi}{\partial y}\, dx = \frac{\sigma^2}{g}\, \phi\, dx.$$

Therefore the integrals along the vertical portions $x = X_0$, $0 > y > -h$, and $x = -X_0$, $0 > y > -h'$ are equal and opposite. By the orthogonality of the expansion of Lemma I only the asymptotic waves contribute to the integral, and

$$\int_{X_0 - ih}^{X_0} \left(\phi \frac{\partial \bar{\phi}}{\partial x} - \bar{\phi} \frac{\partial \phi}{\partial x}\right) dy = 2ik(|b|^2 - |a|^2) \int_{-h}^{0} [\cosh k(y + h)]^2\, dy.$$

Hence the theorem.

Notation: $\{a, b; a', b'\}$ denotes a potential whose asymptotic form on $y = 0$ is $(ae^{ikx} + be^{-ikx})$ as $x \to -\infty$, and $(a'e^{ik'x} + b'e^{-ik'x})$ as $x \to +\infty$.

Theorem III. (1) If $b = 0$ and $a \neq 0$, then a'/a, b'/a are unique.

(2) If $b = 0$ and $a \neq 0$, there is also a solution with $b' = 0$, $a' \neq 0$.

To prove (1):

Since the problem is linear we may take $a = 1$ without loss of generality.

Suppose $\{1, 0; a', b'\}(=\phi)$ and $\{1, 0; a_1', b_1'\}(=\phi_1)$ are two solutions. By linear superposition $a'\phi_1 - a_1'\phi$, i.e. $\{a' - a_1, 0; 0; b_1'a' - b'a_1'\}$ is also a solution. By Th. II

$$|a' - a_1'|^2 kh(1 + \sinh 2kh/2kh)(\cosh k'h')^2$$
$$= -|b_1'a' - b'a_1'|^2 k'h'(1 + \sinh 2k'h'/2k'h')(\cosh kh)^2$$

so that $a' = a_1'$, and, since $a' \neq 0$, $b_1' = b'$.

To prove (2):

Recall that if $\phi(=\{1, 0; a', b'\})$ is a solution of (A), so is $\bar{\phi}(=\{0, 1; \bar{b}', \bar{a}'\})$ By Th. II the determinant

$$\begin{vmatrix} a' & b' \\ \bar{b}' & \bar{a}' \end{vmatrix} = |a'|^2 - |b'|^2$$

is not zero, so that $\bar{a}'\phi - b'\bar{\phi}(= \{\bar{a}', -b'; c, 0\})$ has a non-zero c. (If there is a progressive potential at $-\infty$, there is also one at $+\infty$).

Definition. We define the *right hand reflection coefficient* (R_R) to be the ratio of the amplitudes $|b'|/|a'|$, when in the potential at $-\infty$ $b = 0$, $a \neq 0$ (the potential at $-\infty$ is progressive). By Th. III(1) this ratio is unique.

We define a *left hand reflection coefficient* (R_L) in an analogous manner. By Th. III(2) R_L exists if, and only if, R_R exists. Now R_L and R_R are equal since $R_L = |-b'|/|\bar{a}'|$. Note that this is true generally, i.e. also if the asymptotic depths are different.

2. Two lemmas in the theory of surface waves. In the general reduction of wave problems to an (one dimensional) integral equation we need

Lemma I. (a) If ϕ is bounded and harmonic in $0 > \eta > -h$, $-\infty < \xi < \infty$,

$$\partial\phi/\partial\eta = 0 \quad \text{on} \quad \eta = -h,$$
$$\phi \to 0 \quad \text{as} \quad \xi \to +\infty,$$
$$\sigma^2\phi - g\,\partial\phi/\partial\eta \text{ is } L\text{-integrable on } \eta = 0, \text{ and } O(e^{-c|\xi|}), \text{ for some } c > 0,$$

$$f(x, y) = \frac{1}{2\pi}\int_{-\infty+i\rho}^{\infty+i\rho} \frac{\cosh k(y+h)}{\cosh kh}\,\frac{e^{ikx}\,dk}{\sigma^2 - gk\tanh kh},$$

$$0 < \rho < \min(\pi/2h, c),$$

then

$$\phi(\xi, \eta) = \int_{-\infty}^{\infty}\left[\sigma^2\phi(\xi', 0) - g\frac{\partial\phi}{\partial\eta'}(\xi', 0)\right]f(\xi - \xi', \eta)\,d\xi'.$$

(If $\phi \to 0$ as $\xi \to -\infty$, ρ must lie between 0 and $-\pi/2h$ in the definition of f).

(b) Define $f(x, 0) = f(x)$. If $x > 0$, $-f(x) \geq 0$ and

$$\int_0^{\infty} |f(x)|\,dx = \frac{1}{2\sigma^2}\,\frac{1 - 2kh/\sinh 2kh}{1 + 2kh/\sinh 2kh},$$

$$|f(x)| < \frac{4\exp(\pi x/2h)}{\pi g}\log(1 - \exp\{-\pi x/h\}).$$

$$\text{If } x < 0, \qquad f(x) = f(-x) + i\,\frac{e^{ikx} - e^{-ikx}}{g\tanh kh(1 + 2kh/\sinh 2kh)}.$$

(c) The asymptotic form of $\phi(\xi, 0)$ as $\xi \to -\infty$ is

$$\frac{i}{g\tanh kh(1 + 2kh/\sinh 2kh)}\left\{e^{ik\xi}\int_{-\infty}^{\infty}\left[\sigma^2\phi(\xi', 0)\right.\right.$$

$$\left.- g\frac{\partial\phi}{\partial\eta'}(\xi', 0)\right]e^{-ik\xi'}\,d\xi' - e^{ik\xi}\int_{-\infty}^{\infty}\left[\sigma^2\phi(\xi', 0) - g\frac{\partial\phi}{\partial\eta'}(\xi', 0)\right]$$

$$\left.\cdot e^{ik\xi'}\,d\xi'\right\}.$$

(a) Observe that by Lemma III, Sec. 1, there is at most one potential satisfying conditions (a). We construct its Fourier integral expression.

Since $\sigma^2\phi - g\,\partial\phi/\partial\eta$ is L_1 and $O(e^{-c|\xi|})$ for large $|\xi|$, its Fourier transform $G(k)$ exists in $c - \epsilon \geq \Im(k) \geq -c + \epsilon$, is bounded there, and

$$\phi(\xi, \eta) = (2\pi)^{-1/2} \int_{-\infty+i\rho}^{\infty+i\rho} \frac{G(k)}{\sigma^2 - gk \tanh kh} \frac{\cosh k(\eta + h)}{\cosh kh} e^{ik\xi}\, dk$$

$$c > \rho > 0, \qquad \pi/2h > \rho,$$

satisfies (a).

We convert the Fourier integral into a linear transform of

$$\sigma^2\phi(\xi', 0) - g\frac{\partial\phi}{\partial\eta'}(\xi', 0).$$

We apply the Faltung theorem to

$$G(k) \cosh k(\eta + h)[\cosh kh(\sigma^2 - gk \tanh kh]^{-1}$$

which is permissible since $G(k)$ is bounded on $\Im(k) = \rho$ and the transform is L_2 in $0 \geq \eta \geq -h$.

(b) Consider $x > 0$.

(i) By the theory of residues

$$f(x) = -\sum_1^\infty \frac{e^{-k_r x}}{g \tan k_r h(1 + 2k_r h/\sin 2k_r h)}$$

where ik_r are the imaginary roots of $\sigma^2 = gk \tanh kh$, and $(r - 1/2)\pi < k_r h < r\pi$ (cf. Lemma I of Sec. 1). Now

$$g \tan k_r h(1 + 2k_r h/\sin 2k_r h) = (g/k_r h)[k_r^2 h^2 + (\sigma^2 h/g)(\sigma^2 h/g - 1)]$$

$$> (g/k_r h)[(r - 1/2)^2\pi^2 - 1/4]$$

since $(\sigma^2 h/g)(\sigma^2 h/g - 1)$ attain its minimum at $\sigma^2 h/g = 1/2$.

Thus all the terms of the series are positive and $f(x) < 0$.

Since, further, $k_r h > 1$,

$$g[k_r^2 h + (\sigma^2/g)(\sigma^2 h/g - 1)]/k_r > \frac{1}{4} g\pi(2r - 1) > \frac{1}{4} g\pi r, \qquad \text{and}$$

$$|f(x)| < \frac{4}{\pi g} \sum_{r=1}^{\infty} \exp\{-(2r - 1)\pi x/2h\}/r = \frac{4}{\pi g} \exp\{\pi x/2h\} \log(1 - \exp\{-\pi x/h\}).$$

(ii) Observe that

$$\int_0^\infty |f(x)|\, dx = -\int_0^\infty f(x)\, dx = \int_0^\infty dx\left(\frac{1}{2\pi}\int_{-\infty+i\rho}^{\infty+i\rho} \frac{e^{ikx}}{\sigma^2 - gk \tanh kh}\, dk\right)$$

$$= -\frac{1}{2\pi i}\int_{-\infty+i\rho}^{\infty+i\rho} \frac{dk}{k(\sigma^2 - gk \tanh kh)}$$

where we may invert the order of integration since $|e^{ikx}| = e^{-\rho x}$.

We evaluate the integral by contour integration round a large upper and lower semi-circle with centre $i\rho$. The only singularities in the upper semi-circle are at ik_n and the integral is equal to

$$-g^{-1}\sum_1^\infty [k_r \tan k_r h(1 + 2k_r h/\sin 2k_r h)]^{-1}.$$

The singularities in the lower semi-circle are at 0, $\pm k$ and $-ik_n$, and the integration is clockwise. We get

$$-\Big\{-\sigma^{-2} + 2[gk \tanh kh(1 + 2kh/\sinh 2kh)]^{-1}$$

$$-\sum_1^\infty [gk_r \tan k_r h(1 + 2k_r h/\sin 2k_r h)]^{-1}\Big\}.$$

Equating the two we find

$$-g^{-1}\sum_1^\infty [k_r \tan k_r h(1 + 2k_r h/\sin 2k_r h)]^{-1} = -\frac{1}{2\sigma^2}\frac{1 - 2kh/\sinh 2kh}{1 + 2kh/\sinh 2kh}.$$

(iii) Consider $x < 0$. Integrating clockwise round the lower semi-circle we find

$$f(x) = -\sum_1^\infty \frac{e^{-k_r|x|}}{g \tan k_r h(1 + 2k_r h/\sin 2k_r h)} + i\frac{e^{ikx} - e^{-ikx}}{g \tanh kh(1 + 2kh/\sinh 2kh)}.$$

(c) follows immediately from the fact that in the integral

$$\int_{-\infty}^{\infty}\left(\sigma^2\phi - g\frac{\partial\phi}{\partial\eta'}\right)f(\xi - \xi', 0)\,d\xi'$$

the contribution from the infinite series part of $f(\xi - \xi')$ is small when $|\xi|$ is large: if $|\xi - \xi'|$ is small, $\sigma^2\phi - g\,\partial\phi/\partial\eta'$ is small since $|\xi'|$ is large, and $\sigma^2\phi(\xi', 0) - g\,\partial/\partial\eta'[\phi(\xi', 0)]$ is $O(e^{-c|\xi'|})$; if $|\xi - \xi'|$ is large, the infinite series in f is exponentially small by (b). This proves the lemma.

If the asymptotic depths of the domain of fluid in Fig. 1 are equal, and we transform it conformally into the strip $0 > \eta > -h$, the resulting boundary value problem for the potential $\phi(\xi, \eta)e^{i\sigma t}$ is essentially of the following form:

(B) ϕ bounded and harmonic in $0 > \eta > -h$,
$\partial\phi/\partial\eta = 0$ on $\eta = -h$
$\sigma^2\phi - g\,\partial\phi/\partial\eta = g(\xi)\phi + h(\xi)$ on $\eta = 0$
$\phi \to 0$ as $\xi \to +\infty$, $0 > \eta > -h$.

By the function $f(x, y)$ of Lemma I the two dimensional problem (B) is reduced to the integral equation

$$\phi(\xi, 0) = \int_{-\infty}^{\infty} g(\xi')\phi(\xi', 0)f(\xi - \xi')\,d\xi' + \int_{-\infty}^{\infty} h(\xi')f(\xi - \xi')\,d\xi'.$$

This equation is solved for suitable $g(\xi)$ (suitable obstacles) and integrable $h(\xi)$ by *Lemma II*. If $g(\xi) = O(e^{-c|\xi|})$ and

$$\alpha = [g \tanh kh(1 + 2kh/\sinh 2kh)]^{-1} \max_{\xi} \int_{-\infty}^{\xi} | g(\xi') | | e^{ik(\xi-\xi')} - e^{-ik(\xi-\xi')} | d\xi'$$

$$+ \frac{\max | g(\xi') |}{\sigma^2} \left(\frac{1 - 2kh/\sinh 2kh}{1 + 2kh/\sinh 2kh}\right) < 1$$

the integral equation above can be solved for all ξ by iteration and

$$| \phi | < \frac{\max_{\xi} \left| \int_{-\infty}^{\infty} h(\xi') f(\xi - \xi') \, d\xi' \right|}{1 - \alpha}.$$

Also its solution is unique.

Proof. Write

$$\phi_0 = \int_{-\infty}^{\infty} h(\xi') f(\xi - \xi') \, d\xi', \qquad \phi_{n+1} = \int_{-\infty}^{\infty} \phi_n(\xi', 0) g(\xi') f(\xi - \xi') \, d\xi'.$$

It is readily seen that

$$| \phi_n(\xi, 0) | < \alpha^n \max_{\xi} \left| \int_{-\infty}^{\infty} h(\xi') f(\xi - \xi') \, d\xi' \right|.$$

By dominated convergence it follows that $\sum_0^{\infty} \phi_n(\xi, 0)$ converges to a solution of the integral equation. Further, the solution is unique if ϕ is to be bounded, for the difference Φ between any two solutions satisfies

$$\Phi = \int_{-\infty}^{\infty} \Phi(\xi', 0) g(\xi') f(\xi - \xi') \, d\xi',$$

and therefore

$$| \Phi | \leq \max | \Phi | \int_{-\infty}^{\infty} | g(\xi') f(\xi - \xi') | \, d\xi' \leq \alpha \max | \Phi |.$$

If $\alpha < 1$ this means that $\Phi \equiv 0$.

Thus, if $\alpha < 1$, not only the asymptotic potentials but also the potential in the whole domain of the fluid is defined by the asymptotic potential at one infinity.

3. Reflection of waves by submerged obstacles. In the present section we apply the results of Sec. 2 to calculate how the reflection coefficient depends on the cross sectional shape of the submerged obstacle.

By the usual theory, a potential $\phi e^{-i\sigma t}$ which is progressive at $+\infty$ must satisfy the following conditions:

(C) ϕ bounded and harmonic in the domain of the fluid,
$\sigma^2 \phi - g \, \partial\phi/\partial y = 0$ on $y = 0$,
$\partial\phi/\partial n = 0$ on the lower boundary,
$\phi \to a e^{ikx} \cosh k(y + h)/\cosh kh$ as $x \to + \infty$.

We map the strip $0 > \eta > -h$ on the domain of the fluid (so that the infinities correspond) by the conformal transformation $z(\zeta)(\zeta = \xi + i\eta)$ which is unique except for a linear shift in ξ. The boundary conditions then become

(C′) ϕ bounded and harmonic in $0 > \eta > -h$,
$\sigma^2\phi - g\,\partial\phi/\partial\eta + \sigma^2(dz/d\zeta - 1)\phi = 0$ on $\eta = 0$,
$\partial\phi/\partial\eta = 0$ on $\eta = -h$,
$\phi \to ae^{ik\xi}\cosh k(\eta + h)/\cosh kh$ as $\xi \to +\infty$.

Theorem I. There is a unique ϕ satisfying C' provided (e.g.)

$$\alpha = \frac{2k}{1 + 2kh/\sinh 2kh}\int_{-\infty}^{\infty}\left|\frac{dz}{d\zeta} - 1\right|_{\eta=0} d\zeta + \max_{\eta=0}\left|\frac{dz}{d\zeta} - 1\right|\frac{1 - 2kh/\sinh 2kh}{1 + 2kh/\sinh 2kh} < 1. \tag{3.1}$$

ϕ can then be computed by iteration, and the reflection coefficient R is given by

$$\frac{R_0 - \alpha^2/2(1-\alpha)}{\{1 + [R_0 - \alpha^2/2(1-\alpha)]^2\}^{1/2}} < R < \frac{R_0 + \alpha^2/2(1-\alpha)}{\{1 + [R_0 + \alpha^2/2(1-\alpha)]^2\}^{1/2}},$$

where

$$R_0 = \frac{k}{1 + 2kh/\sinh 2kh}\left|\int_{-\infty}^{\infty}\left(\frac{dz}{d\zeta} - 1\right)e^{2ik\xi}\,d\zeta\right|.$$

Proof. Write

$$\phi = \phi_1 + ae^{ik\xi}\cosh k(\eta + h)/\cosh kh.$$

Then the conditions for ϕ_1 are of the form (B) of Sec. 2, if $-\sigma^2(dz/d\zeta - 1)$ is substituted for $g(\xi)$, and $-\sigma^2(dz/d\zeta - 1)ae^{ik\xi}$ for $h(\xi)$. By Lemma II of Sec. 2 ϕ_1 can be calculated by iteration if
$|\,dz/d\zeta - 1\,| = O(e^{-\epsilon|\xi|})$, and the α of Lemma II, Sec. 2 is less than 1.
These conditions are checked in (i) and (ii).
To get bounds for the reflection coefficient we get in (iii) bounds in terms of α for the reflected wave.

(i) $$\left|\frac{dz}{d\zeta} - 1\right| = O(\exp\{-\pi\,|\,\xi\,|/h\}).$$

Consider the inverse $\zeta(z)$ of $z(\zeta)$, for definiteness in $x < -X_0$. The imaginary part of $\zeta(z)$ is harmonic and bounded in $0 > y > -h$, $x < -X_0$. Also $\Im[\zeta(z) - z] = 0$ on $y = 0$ and $y = -h$ for $x < -X_0$. Thus by the reflection principle ζ can be continued across $y = 0$ and $y = -h$, and is therefore analytic on $0 \geq y \geq -h$, $x = x_1$ if $x_1 < -X_0$. We can therefore expand

$$\Im[\zeta(z) - z] = \sum_1^{\infty} f_n(x)\sin(n\pi y/h)$$

for $x < -X_0$ where

$$f_n(x) = 2/(\pi h)\int_{-h}^{0}[\eta(x, y) - y]\sin(n\pi y/h)\,dy.$$

By repeated integration by parts

$d^2f_n/dx^2 = (n\pi/h)^2 f_n$

and hence

$f_n = a_n e^{n\pi z/h} + b_n e^{-n\pi z/h}$.

Since $\eta(x, y) - y$ is bounded, $b_n = 0$ and $a_n e^{-n\pi X_0/h}$ is bounded. Thus

$$\left| \frac{d\zeta}{dz} - 1 \right| \leq \sum_1^\infty (n\pi/h) a_n \exp\{n\pi x/h\} = O(\exp\{-\pi(|x| - X_0)/h\}), \text{ and } \quad \zeta(z) \sim z.$$

Similarly for $x > +X_0$.

(ii) It is readily verified that (3.1) is a bound for α of Sec. 2.

(iii) By Lemma I of Sec. 2 the reflected wave is asymptotically

$$e^{-ik\xi}[g \tanh kh(1 + 2kh/\sinh 2kh)]^{-1} \int_{-\infty}^{\infty} \left(\sigma^2\phi - g\frac{\partial\phi}{\partial\eta'}\right) e^{ik\xi'}\, d\xi'$$

$$= e^{-ik\xi}\sigma^2[g \tanh kh(1 + 2kh/\sinh 2kh)]^{-1} \int_{-\infty}^{\infty} \phi\left(\frac{dz}{d\zeta'} - 1\right) e^{ik\xi'}\, d\xi'.$$

If

$$\phi_1 = \sum_1^\infty \phi_n^{(1)},$$

$$|\phi_n^{(1)}| < a\alpha^n \quad \text{and} \quad |\phi_1| < a\alpha/(1 - \alpha),$$

and the amplitude of the reflected wave differs from

$$ak(1 + 2kh/\sinh 2kh)^{-1} \int_{-\infty}^{\infty} \left(\frac{dz}{d\zeta'} - 1\right) e^{2ik\xi'}\, d\xi'$$

by less than

$$ak\alpha(1 - \alpha)^{-1}(1 + 2kh/\sinh 2kh)^{-1} \int_{-\infty}^{\infty} \left| \frac{dz}{d\zeta'} - 1 \right| d\xi'.$$

Since

$$k(1 + 2kh/\sinh 2kh)^{-1} \int_{-\infty}^{\infty} \left| \frac{dz}{d\zeta'} - 1 \right| d\xi' < \frac{1}{2}\alpha,$$

by Th. II of Sec. 1 the incoming wave lies between $a\{1 + [R_0 \pm \alpha^2/2(1 - \alpha)]^2\}^{1/2}$ and

$$\frac{R_0 - \alpha^2/2(1 - \alpha)}{\{1 + [R_0 - \alpha^2/2(1 - \alpha)]^2\}^{1/2}} < R < \frac{R_0 + \alpha^2/2(1 - \alpha)}{\{1 + [R_0 + \alpha^2/2(1 - \alpha)]^2\}^{1/2}}.$$

Example. Consider a vertical barrier of height r in the bottom of a canal of depth h. Let $r\pi/h = \epsilon$. Then

$$\alpha = 8kh[\pi(1 + 2kh/\sinh 2kh)]^{-1} \log \sec(r\pi/2h) + 2\sec^2(r\pi/4h)\frac{1 - 2kh/\sinh 2kh}{1 + 2kh/\sinh 2kh}$$

$$= O(r/h)^2,$$

and, as $kh \to 0$

$$R = (2/\pi)kh \log \sec(r\pi/2h) + O(r/h)^4.$$

Proof. Without loss of generality we take $h = \pi$ so that the height of the reef is ϵ.

(i) The transformation function is obtained by Schwarz-Christoffel:

$$\frac{dz}{d\zeta} = \frac{e^{\zeta} + 1}{\{(e^{\zeta} + \rho)(e^{\zeta} + \rho^{-1})\}^{1/2}}$$

(ii) ρ is determined from the height of the reef:
since $\log \rho - i\pi$ is the image of the foot, $-i\pi$ of the tip of the barrier

$$i\epsilon = \int_{\log \rho - i\pi}^{-i\pi} \frac{dz}{d\zeta}\, d\zeta = \int_{\log \rho}^{0} \frac{1 - e^{\xi}}{\{(e^{\xi} - \rho)(\rho^{-1} - e^{\xi})\}^{1/2}}\, d\xi = i2 \tan^{-1} \frac{1 - \rho}{2\rho^{1/2}}.$$

(iii) Since $0 < dz/d\zeta < 1$, on $\eta = 0$, $\quad |\, 1 - dz/d\zeta\, |_{\eta=0} = 1 - dz/d\zeta$

so that $$\int_{-\infty}^{\infty} \left| 1 - \frac{dz}{d\zeta} \right| d\zeta = \int_{-\infty}^{\infty} \left(1 - \frac{dz}{d\zeta}\right) d\zeta = 4 \log \frac{1 + \rho}{2\rho^{1/2}}.$$

(iv) By (ii) $$4 \log \frac{1 + \rho}{2\rho^{1/2}} = 4 \log \sec \frac{1}{2}\epsilon.$$

(v) It is easily verified that $dz/d\zeta$ attains its minimum at $\zeta = 0$ so that

$$\max_{\eta=0} \left| 1 - \frac{dz}{d\zeta} \right| = 2 \sin^2 \frac{1}{4}\epsilon.$$

Thus we get α, and R as in the discussion of the paper given above.

To compare the α for various domains of fluid we use the following lemma.

Lemma. Suppose $\zeta(z)$ maps a domain D, which is bounded above by $y = 0$ and is contained in the strip $0 > y > -h$, into the strip $0 > \eta > -h$ so that the infinities correspond. Then

$$1 \leq \left(\frac{d\zeta}{dz}\right)_{y=0} = \frac{\partial \xi}{\partial x} < \infty.$$

For, $\Im[\zeta(z) - z]$ is bounded and harmonic in D, zero on $y = 0$ and not positive on the lower boundary of D. Thus $\Im[\zeta(z) - z] \leq 0$ in D.

Since $$\Im[\zeta(x, 0)] = 0, \qquad \frac{\Im[\zeta(x, -y) - \zeta(x, 0)] + y}{-y} \geq 0,$$

and therefore $$\lim_{y \to 0} \frac{\Im[\zeta(x_1 - y) - \zeta(x, 0)]}{-y} - 1 \geq 0 \qquad \text{i.e.} \qquad \frac{\partial}{\partial y} \Im[\zeta(z)]_{y=0} \geq 1.$$

By Cauchy's relation $(\partial/\partial y)\Im = (\partial/\partial x)\Re$ so that $\partial\xi/\partial x \geq 1$ on $y = 0$, $dz/d\zeta \neq 0$ since the transformation can be continued across $y = 0$ and is conformal.

Theorem II (cf. Fig. 2). If two domains D and D_1 are both bounded above by $y = 0$ and are contained in the strip $0 > y > -h$, and if $D_1 \subset D$

then $$\alpha \leq \alpha_1 .$$

Let $\zeta_1(z)$ map D_1 on $0 > \eta > -h$, and without loss of generality we may take $\zeta(0) = \zeta_1(0) = 0$.

It is sufficient to show that

$$\max_{\eta=0}\left|1-\frac{dz}{d\zeta}\right| \leq \max_{\eta=0}\left|1-\frac{dz_1}{d\zeta}\right| \quad \text{and} \quad \int_{-\infty}^{\infty}\left|1-\frac{dz}{d\zeta}\right| d\xi \leq \int_{-\infty}^{\infty}\left|1-\frac{dz_1}{d\zeta}\right| d\xi.$$

Suppose $\zeta(D_1) = D'$. Since $D_1 \subset D$, D' is contained in the strip $0 > y > -h$. If $\zeta^{(1)}(u)$ maps D' on the strip, $\zeta_1(z) = \zeta^{(1)}[\zeta(z)]$ and by the lemma $d\zeta^{(1)}/du \geq 1$ when $\Im(u) = 0$. Therefore

$$\frac{d\zeta_1}{dz} = \frac{d\zeta^{(1)}}{d\zeta}\left(\frac{d\zeta}{dz}\right)_{\eta=0} \geq \frac{d\zeta}{dz}$$

and

$$\max_{\eta=0}\left(1-\frac{dz}{d\zeta}\right) \leq \max_{\eta=0}\left(1-\frac{dz_1}{d\zeta}\right). \tag{3.2}$$

So far we have used the condition that D_1 is contained in D, but not that D is contained in $0 > y > -h$. Then $1 - (dz/d\zeta)_{\eta=0} \geq 0$ and

$$\max_{\eta=0}\left|1-\frac{dz}{d\zeta}\right| \leq \max_{\eta=0}\left|1-\frac{dz_1}{d\zeta}\right|.$$

Next consider the integrals

$$\int_{-\infty}^{\infty}\left|1-\frac{dz}{d\zeta}\right| d\xi = \int_{-\infty}^{\infty}\left(1-\frac{dz}{d\zeta}\right) d\xi = \int_{-\infty}^{\infty}\left(\frac{d\xi}{dz}-1\right) dz$$

$$= \int_{-\infty}^{\infty}\left|\frac{d\xi}{dz}-1\right| dz \leq \int_{-\infty}^{\infty}\left|\frac{d\xi_1}{dz}-1\right| dz = \int_{-\infty}^{\infty}\left|1-\frac{dz_1}{d\zeta}\right| d\xi.$$

Hence the theorem.

Theorem III (cf. Fig. 3). Suppose a domain D is bounded above by $y = 0$. Denote its intersection with the strip $0 > y > -h$ by D_1, its join with the strip by D_2. Then

$$\max_{\eta=0}\left|1-\frac{dz}{d\zeta}\right| \leq \max_{\eta=0}\left(1-\frac{dz_1}{d\xi}\,\frac{dz_2}{d\xi}-1\right) \leq \max_{\eta=0}\left|1-\frac{dz_1}{d\xi}\right| + \max_{\eta=0}\left|\frac{dz_2}{d\xi}-1\right|,$$

and $$\int_{-\infty}^{\infty}\left|1-\frac{dz}{d\zeta}\right| d\xi \leq \int_{-\infty}^{\infty}\left|1-\frac{dz_1}{d\zeta}\right| d\xi + \int_{-\infty}^{\infty}\left|\frac{dz_2}{d\zeta}-1\right| d\xi.$$

(i) By (3.2) $\max_{\eta=0}(1 - dz/d\zeta) \leq \max_{\eta=0}(1 - dz_1/d\zeta)$, since $D_1 \subset D$ and $\max_{\eta=0}(dz/d\zeta - 1) \leq \max_{\eta=0}(dz_2/d\zeta - 1)$ since $D \subset D_2$. Because on $\eta = 0$ $dz_1/d\zeta \leq 1$, $dz_2/d\zeta \geq 1$ the first result follows.

(iii) Since $D_1 \subset D \subset D_2$, by (3.2) $d\zeta_2(x, 0)/dz \leq d\zeta(x, 0)/dz \leq d\zeta(x_1, 0)/d\zeta$.

Thus $\left|\frac{d\xi}{dz}-1\right|_{\eta=0} \leq \left|\frac{d\xi_1}{dz}-1\right|$ when $\frac{d\xi}{dz} \geq 1$.

and $\left|\frac{d\xi}{dz}-1\right|_{\eta=0} \leq \left|\frac{d\xi_2}{dz}-1\right|$ when $\frac{d\xi}{dz} \leq 1$ so that

$$\left|\frac{d\xi}{dz}-1\right|_{\eta=0} \leq \left|\frac{d\xi_1}{dz}-1\right| + \left|\frac{d\xi_2}{dz}-1\right|.$$

Hence

$$\int_{-\infty}^{\infty} \left| 1 - \frac{dz}{d\zeta} \right| d\zeta = \int_{-\infty}^{\infty} \left| \frac{d\zeta}{dz} - 1 \right| dz \leq \int_{-\infty}^{\infty} \left| \frac{d\zeta_1}{dz} - 1 \right| dz + \int_{-\infty}^{\infty} \left| \frac{d\zeta_2}{dz} - 1 \right| dz,$$

and the second result follows.

4. Reflection from obstacles in the surface. In the study of the reflection of waves by obstacles in the surface we consider first a special case: reflection by a flat plate lying in the surface. By change of scale we make the half width of the plate unity.

Theorem I. Provided

$$4 \sin^2 k/(1 + 2kh/\sinh 2kh) + \frac{1 - 2kh/\sinh 2kh}{1 + 2kh/\sinh 2kh} < 1 \qquad \text{or}$$

$$4 \sin^2 k/(1 + 2kh/\sinh 2kh) + 2[2^{-1/2} \exp(-2^{1/2}kh) + (kh)^{-1} + 2\pi^{-1/2}k^{1/2}) < 1,$$

there is a unique potential ϕ, bounded in $0 > y > -h$ which satisfies:

(D) $\sigma^2\phi - g\,\partial\phi/\partial y = 0 \qquad y = 0 \qquad |x| > 1,$

$\partial\phi/\partial y = 0 \qquad y = 0 \qquad |x| < 1, \qquad$ and on $y = -h,$

$\phi \to ae^{ikx} \cosh k(y + h)/\cosh kh \qquad$ as $\qquad x \to +\infty, \qquad 0 > y > -h.$

Also

$$R \sim \frac{2k}{1 + 2kh/\sinh 2kh} + O(k^2),$$

and for $0 < \alpha^2/(1 - \alpha) < \sin 2k/2k$ we get precise bounds

$$\frac{2k[\sin 2k/2k - \alpha^2/(1 - \alpha)]/(1 + 2kh/\sinh 2kh)}{\{1 + 4k^2[\sin 2k/2k - \alpha^2/(1 - \alpha)]^2/(1 + 2kh/\sinh 2kh)^2\}^{1/2}}$$

$$< R < \frac{2k[\sin 2k/2k + \alpha^2/(1 - \alpha)]/(1 + 2kh/\sinh 2kh)}{\{1 + 4k^2[\sin 2k/2k + \alpha^2/(1 - \alpha)]^2/(1 + 2kh/\sinh 2kh)^2\}^{1/2}}. \tag{4.1}$$

(i) Observe first that if $f(x)$ is the function of Lemma I, Sec. 2, and $\alpha = \sigma^2 \int_{-1}^{1} |f(x - x')|\,dx' < 1$ in $-1 < x < 1$, ϕ is unique.
By that lemma the difference Φ of any two potentials satisfying (D) also satisfies

$$\Phi(x, 0) = \int_{-1}^{1} \left(\sigma^2\Phi - g\frac{\partial\Phi}{\partial y'}\right) f(x - x')\,dx' = \sigma^2 \int_{-1}^{1} \Phi f(x - x')\,dx',$$

so that

$$|\Phi(x, 0)|_{|x|<1} \leq \alpha \max_{|x'|<1} |\Phi(x', 0)|.$$

Thus $\Phi(x', 0) = 0$ in $|x'| < |,$ also $\partial\Phi/\partial y' = 0$

so that $\Phi \equiv 0$.

(ii) To construct a ϕ we use the method of Lemma II, Sec. 2.

If $\phi = \phi_1 + ae^{ikx} \cosh k(y + h)/\cosh kh$, then ϕ_1 satisfies conditions of form (B):

$g(x) = 0, \mid x \mid > 1; \quad g(x) = -\sigma^2, \mid x \mid < 1;$

$$h(x) = 0, \mid x \mid > 1, h(x) = -a\sigma^2 e^{ikx}, \mid x \mid < 1.$$

The iteration converges in $\mid x \mid < 1$ if $\sigma^2 \int_{-1}^{1} f(x - x')dx' < 1$ for $\mid x \mid < 1$ to give $\phi_1(x, 0), \mid x \mid < 1$; and since

$$g \frac{\partial \phi_1}{\partial y}(x, 0) = -agk \tanh kh\, e^{ikx} = -a\sigma^2 e^{ikx}$$

on the plate, $\sigma^2\phi_1 - g\, \partial\phi_1/\partial y$ is known on $y = 0$, and by Lemma I, Sec. 2, we get ϕ_1 in the whole strip.

(iii) Write

$$\phi_1 = \sum_0^\infty \phi_n^{(1)} \qquad \text{with} \qquad \phi_0^{(1)} = \sigma^2 a \int_{-1}^{1} e^{ikx'} f(x - x', y)\, dx',$$

$$\phi_{n+1} = \sigma^2 \int_{-1}^{1} \phi_n^{(1)}(x', 0) f(x - x', y)\, dx'.$$

The asymptotic form of $\phi_0^{(1)}(x, y)$, (at $-\infty$) is

$$\frac{aki}{1 + 2kh/\sinh 2kh} \left[e^{-ikx} \int_{-1}^{1} e^{2ikx'}\, dx' - e^{ikx} \int_{-1}^{1} dx' \right] \cosh k(y + h)/\cosh kh,$$

and the asymptotic form of $\phi_{n+1}^{(1)}$ is

$$\frac{ki}{1 + 2kh/\sinh 2kh} \left[e^{-ikx} \int_{-1}^{1} \phi_n^{(1)} e^{ikx'}\, dx' \right.$$

$$\left. - e^{ikx} \int_{-1}^{1} \phi_n^{(1)} e^{-ikx'}\, dx' \right] \cosh k(y + h)/\cosh kh.$$

The reflected wave [i.e. coefficient of $e^{-ikx} \cosh k(y + h)/\cosh kh$ in the asymptotic form] of

$$\sum_1^\infty \phi_n^{(1)}$$

is less than

$$\frac{k}{1 + 2kh/\sinh 2kh} \sum_1^\infty \int_{-1}^{1} \mid \phi_n^{(1)} \mid dx' < \frac{2ka}{1 + 2kh/\sinh 2kh} \sum_1^\infty \alpha^{n+1}$$

$$= \frac{2ka}{(1 + 2kh/\sinh 2kh)} \frac{\alpha^2}{(1 - \alpha)}.$$

The reflected wave in $\phi_0^{(1)}$ is $a \sin 2k/(1 + 2kh/\sinh 2kh)$.

The amplitude of the reflected wave of

$$\sum_0^\infty \phi_n^{(1)}$$

lies between

$$\frac{2ka}{1 + 2kh/\sinh 2kh}\left[\frac{\sin 2k}{2k} \pm \frac{\alpha^2}{1 - \alpha}\right].$$

Since the transmitted wave is $ae^{ikx} \cosh k(y + h)/\cosh kh$, we get a bound for the in coming wave, and find (4.1).

Evaluation of α.

(iv) First we get a bound suitable for small kh. Denote the infinite series part of $f(x)$ by $f_1(x)$. By Lemma I, Sec. 2,

$$\int_{-1}^{1} | f(x - x') | dx' < 2\int_{0}^{1} | f_1(x') | dx'$$

$$+ [g \tanh kh(1 + 2kh/\sinh 2kh)]^{-1} \int_{x}^{1} | e^{ik(x-x')} - e^{-ik(x-x')} | dx'.$$

Now

$$\int_{0}^{1} | f_1(x') | dx' < \int_{0}^{\infty} | f_1(x') | dx' = \frac{1}{2\sigma^2} \frac{1 - 2kh/\sinh 2kh}{1 + 2kh/\sinh 2kh}.$$

Also

$$\int_{x}^{1} | e^{ik(x-x')} - e^{-ik(x-x')} | dx' = 2\int_{0}^{1-x} | \sin ku | du \leq 2\int_{0}^{2} | \sin ku | du$$

$$= 4 \sin^2 k/k \qquad \text{if} \qquad k < \frac{1}{2}\pi.$$

Thus

$$\sigma^2 \int_{-1}^{1} | f(x - x') | dx' < 4 \sin^2 k/(1 + 2kh/\sinh 2kh) + \frac{1 - 2kh/\sinh 2kh}{1 + 2kh/\sinh 2kh}.$$

(v) If $kh \to \infty$ the bound becomes useless since the second summand tends to 1. We use a refined estimate for $\int_0^1 | f_1(x') | dx'$. Recall that

$$- f_1(x) = \sum_{1}^{\infty} \frac{e^{-k_r x}}{g \tan k_r h(1 + 2k_r h/\sin 2k_r h)} < \frac{1}{g} \sum_{1}^{\infty} \frac{(hk_r)e^{-k_r x}}{(2r - 1)^2\pi^2/4 + \tau}$$

where

$$\tau = (\sigma^2 h/g)(\sigma^2 h/g - 1).$$

Thus

$$\int_{0}^{1} | f_1(x) | dx < \frac{h}{g} \sum_{1}^{\infty} \frac{(1 - e^{-k_r})}{(2r - 1)^2\pi^2/4 + \tau}$$

$$= \frac{h}{2g} \sum_{-\infty}^{\infty} \frac{1}{(2r - 1)^2\pi^2/4 + \tau} - \frac{h}{g} \sum_{1}^{\infty} \frac{e^{-k_r}}{(2r - 1)^2\pi^2/4 + \tau} > 0.$$

By contour integration of $\cot \pi z[(2z-1)^2/4+\tau]^{-1}$ around a large rectangle it is seen that

$$\sum_{-\infty}^{\infty} \frac{1}{(2r-1)^2\pi^2/4+\tau} = \frac{\tanh \tau^{1/2}}{\tau^{1/2}}.$$

$$\sum_{1}^{\infty} \frac{e^{-k\tau}}{(2r-1)^2\pi^2/4+\tau} > \sum_{1}^{\infty} \frac{e^{-\tau\pi/h}}{(2r-1)^2\pi^2/4+\tau^2} > \int_1^{\infty} \frac{e^{-\pi x/h}}{(2r-1)^2\pi^2/4+\tau^2}\,dx,$$

since $k,h < \pi\tau$, and the terms of the series are decreasing.

$$\int_1^{\infty} \frac{e^{-\pi x/h}}{(2x-1)^2\pi^2/4+\tau^2}\,dx = \frac{1}{\pi\tau^{1/2}} \int_{\pi/2\tau^{1/2}}^{\infty} \frac{\exp(-\tau^{1/2}u/h)}{u^2+1}\,du$$

$$> \frac{1}{\pi\tau^{1/2}} \int_0^{\infty} \frac{du}{u^2+1} - \frac{1}{\pi\tau^{1/2}} \int_0^{\pi/2\tau^{1/2}} du - \frac{1}{\pi\tau^{1/2}} \left\{ \int_0^N + \int_N^{\infty} \frac{1-\exp(-\tau^{1/2}u/h)}{u^2+1}\,du \right.$$

$$= \frac{1}{2\tau^{1/2}} - \frac{1}{2\tau} - \frac{1}{\pi\tau^{1/2}} \left\{ \int_0^N + \int_N^{\infty} \frac{1-\exp(-\tau^{1/2}u/h)}{u^2+1}\,du \right\}.$$

Since

$$1-\exp(-\tau^{1/2}N/h) < \tau^{1/2}N/h, \qquad \frac{1}{\pi\tau^{1/2}} \int_0^N \frac{1-\exp(-\tau^{1/2}u/h)}{u^2+1}\,du < \frac{1}{2}N/h, \quad \text{and}$$

$$\int_N^{\infty} \frac{1-\exp(-\tau^{1/2}u/h)}{u^2+1}\,du < \int_N^{\infty} \frac{du}{u^2+1} = \frac{1}{2}\pi - \tan^{-1} N = \tan^{-1}\frac{1}{N} < \frac{1}{N}.$$

Therefore

$$\frac{1}{\pi\tau^{1/2}} \int_0^{\infty} \frac{1-\exp(-\tau^{1/2}uh)}{u^2+1}\,du < \frac{1}{\pi N\tau^{1/2}} + \frac{1}{2}\frac{N}{h}.$$

If we choose

$$N = (2h/\pi)^{1/2}\tau^{-1/4}, \qquad \int_0^1 |f_1(x)|\,dx < \frac{h}{g}\left[\frac{\tanh \tau^{1/2}-1}{2\tau^{1/2}} + \frac{1}{2\tau} + \left(\frac{2}{\pi h}\right)^{1/2}\tau^{1/4}\right].$$

Since $(\tanh \tau^{1/2}-1)/2 = \exp(-2\tau^{1/2})/[1+\exp(-2\tau^{1/2})] < \exp(-2^{1/2}kh)$, and $(1/h\tau^{1/2})^{1/2} < 2^{1/2}h^{-1}k^{-1/2}$ we get after some simplification that

$$2\sigma^2 \int_0^1 |f_1(x)|\,dx < 2[2^{-1/2}\exp(-2^{1/2}kh) + (kh)^{-1} + 2\pi^{-1/2}k^{1/2}].$$

Since at the edges of the plate in Th. I the horizontal velocity gets infinite while in the derivation of the boundary conditions the velocity is assumed to be everywhere small, it is not clear that the calculation gives a sensible approximation to the reflection of waves by shallow obstacles. We shall therefore show separately that the reflection of an arbitrary (cylindrical) obstacle of beam equal to the width of the plate tends to the reflection calculated in Th. I provided only that the draught tends to zero.

Theorem II. We map the domain of the fluid on $0 > \eta > -h$ so that the edges of the obstacle are at $(\xi = \pm 1, \eta = 0)$. If

(D′) ϕ is bounded and harmonic in $0 > \eta > -h$,

$$\partial\phi/\partial\eta = 0 \quad \text{on} \quad \eta = -h, \quad \text{and on} \quad \eta = 0, \quad |\xi| < 1,$$

$$\sigma^2\phi - g\,\partial\phi/\partial\eta + \sigma^2(dz/d\zeta - 1)\phi = 0 \quad \text{on} \quad \eta = 0 \quad |\xi| > 1,$$

$$\phi \to ae^{ik\xi}\cosh k(\eta + h)/\cosh kh \quad \text{as} \quad \xi \to \infty,$$

ϕ can be constructed by iteration provided

$$0 < \delta = \frac{\beta}{1 - \alpha} < 1 \qquad \text{where}$$

$$\sigma^2 \int_{-1}^{1} |f(\xi - \xi')|\, d\xi' \leq \alpha$$

$$\sigma^2\left(\int_{-\infty}^{-1} + \int_{1}^{\infty}\left|\frac{dz}{d\zeta'} - 1\right|\right)\left|f(\xi - \xi')\right| d\xi'\Big) \leq \beta \qquad \text{for all } \xi.$$

Further if the draught is small, β is small.

(i) Let ϕ_0 be the potential of Th. I. We write then

$$\phi = \phi_0 + \sum_{1}^{\infty} \phi_n ,$$

and split up ϕ_{n+1} into $\phi_{n+1}^{(1)} + \phi_{n+1}^{(2)}$, which satisfy the following conditions:

(D″) $\phi_{n+1}^{(1)}$ is bounded and harmonic in $0 > \eta > -h$,

$$\partial\phi_{n+1}^{(1)}/\partial\eta = 0 \quad \text{on} \quad \eta = -h,$$

$$\sigma^2\phi_{n+1}^{(1)} - g\,\partial\phi_{n+1}^{(1)}/\partial\eta = \begin{cases} -\sigma^2(dz/d\zeta - 1)\phi_n & \text{on} \quad \eta = 0, \quad |\xi| > 1, \\ 0 & \text{on} \quad \eta = 0, \quad |\xi| < 1, \end{cases}$$

$$\phi_{n+1}^{(1)} \to 0 \quad \text{as} \quad \xi \to +\infty.$$

(D‴) $\phi_{n+1}^{(2)}$ is bounded and harmonic in $0 > \eta > -h$,

$$\partial\phi_{n+1}^{(2)}/\partial\eta = 0 \quad \text{on} \quad \eta = -h, \quad \partial\phi_{n+1}^{(2)}/\partial\eta = -\partial\phi_{n+1}^{(1)}/\partial\eta = -(\sigma^2/g)\phi_{n+1}^{(1)}$$

$$\text{on} \quad \eta = 0 \quad |\xi| < 1,$$

$$\sigma^2\phi_{n+1}^{(2)} - g\,\partial\phi_{n+1}^{(2)}/\partial\eta = 0 \quad \text{on} \quad \eta = 0 \quad |\xi| > 1,$$

$$\phi_{n+1}^{(2)} \to 0 \quad \text{as} \quad \xi \to +\infty.$$

Let $|\phi_{n+1}^{(1)}| < M_{n+1}^{(1)}$, $|\phi_{n+1}^{(2)}| < M_{n+1}^{(2)}$ and $|\phi_{n+1}| < M_{n+1}$.

(ii) By Lemma II, Sec. 2 (we may take) $M_{n+1}^{(1)} < \beta M_n$.

$$\phi_{n+1}^{(2)} = \sigma^2 \int_{-1}^{1} \phi_{n+1}^{(2)} f(\xi - \xi')\, d\xi' - g \int_{-1}^{1} \frac{\partial \phi_{n+1}^{(2)}}{\partial \eta'} f(\xi - \xi')\, d\xi'$$

$$= \sigma^2 \int_{-1}^{1} \phi_{n+1}^{(2)} f(\xi - \xi')\, d\xi' + \sigma^2 \int_{-1}^{1} \phi_{n+1}^{(1)} f(\xi - \xi')\, d\xi' .$$

Therefore

$$M_{n+1}^{(2)} < \alpha M_{n+1}^{(2)} + \alpha M_{n+1}^{(1)} ,$$

so that

$$M_{n+1}^{(2)} < \frac{\alpha\beta M_n}{1 - \alpha} \qquad \text{and} \qquad M_{n+1} < \frac{\beta M_n}{1 - \alpha} = \delta M_n .$$

(iii) If $\delta < 1$, $\alpha < 1$ so that ϕ_0 exists and the iteration converges to a solution of (D′). If δ is small compared with α, the solution is nearly equal to ϕ_0 .

(iv) $\delta \to 0$ if the draught is small, provided at its edges A, B the obstacle has tangents which make angles greater than ϵ with the mean free surface (Fig. 5). We take A at $(-1, 0)$.

We enclose the obstacle in a trapezoid $ABCD$, and denote by D_1 the domain obtained by removing the trapezoid from the strip $0 > y > -h$. Let $z_1(\zeta)$ map $0 > \eta > -h$ on D_1 , keeping A fixed, and let its inverse be $\zeta_1(z)$. Assume for the moment that (i) $| \geq d\zeta/dz \geq d\zeta_1/dz > 0$ on $z < -1$; (ii) in $z \leq 1 - \rho$, $\rho > 0$, $1 - d\zeta_1/dz$ tends to 0 uniformly as the distance d between AB and $CD \to 0$, ϵ fixed;
(iii) $d\zeta_1/dz$ decreases monotonically to zero in $-\infty < z < -1$ as $z \to -1$ so that if z, z', $z' + h$ are less than -1 and $z < z'$

$$| \zeta_1(z' + h) - \zeta_1(z') | < | \zeta_1(z + h) - \zeta_1(z) | .$$

Now

$$\int_{-\infty}^{-1} | f(\xi - \xi') | \left| \frac{dz}{d\zeta'} - 1 \right| d\zeta'$$

$$\leq [g \tanh kh(1 + 2kh/\sinh 2kh)]^{-1} \int_{-\infty}^{-1} \left| \frac{dz}{d\zeta'} - 1 \right| | e^{ik(\xi - \xi')} - e^{-ik(\xi - \xi')} | \, d\xi'$$

$$+ \int_{-\infty}^{-1} | f_1(\xi - \xi') | \left| \frac{dz}{d\zeta'} - 1 \right| d\xi' = I_1 + I_2(\text{p.def.})$$

$$I_1 < 2[g \tanh kh(1 + 2kh/\sinh 2kh)]^{-1} \int_{-\infty}^{-1} \left| \frac{dz}{d\zeta'} - 1 \right| d\zeta'$$

$$= 2[g \tanh kh(1 + 2kh/\sinh 2kh)]^{-1} \int_{-\infty}^{-1} \left| \frac{d\zeta}{dz} - 1 \right| dz$$

$$< 2[g \tanh kh(1 + 2kh/\sinh 2kh)]^{-1} \int_{-\infty}^{-1} \left| \frac{d\zeta_1}{dz} - 1 \right| dz, \text{ by (i)}, = J_1(\text{p.def.}).$$

A simple calculation shows that $J_1 \to 0$ as $d \to 0$.

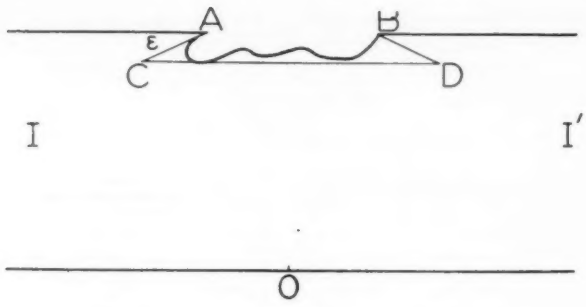

FIG. 5.

Let $\xi = \xi(z)$. Then, since $|f_1(x)|$ is decreasing, and, by (i), $|\xi_1(z) - \xi_1(z')| \leq |\xi(z) - \xi(z')|$

$$I_2 = \int_{-\infty}^{-1} |f_1[\xi(z) - \xi(z')]| \left| \frac{d\zeta}{dz'} - 1 \right| dz' < \int_{-\infty}^{-1} |f_1[\xi_1(z) - \xi_1(z')]| \left| \frac{d\zeta_1}{dz'} - 1 \right| dz',$$

$$< \left(\int_{-\infty}^{-1+\rho} |f_1[\xi_1(z) - \xi_1(z')]| \, dz' \right) \left| \frac{d\zeta_1(-1+\rho, 0)}{dz} - 1 \right|$$

$$+ \int_{-1+\rho}^{-1} |f_1[\xi_1(z) - \xi_1(z')]| \, dz', \qquad \text{by (i) and (ii)},$$

$$< 2\left(\int_{-\infty}^{-1} |f_1[-1 - \xi_1(z')]| \, dz' \right) \left| \frac{d\zeta_1(-1+\rho, 0)}{dz} - 1 \right|$$

$$+ 2 \int_{-1+1/2\rho}^{-1} |f_1[-1 - \xi_1(z')]| \, dz', \qquad \text{by (iii)}, \qquad \text{independently of } z.$$

Note that both integrals decrease if ρ is fixed and d decreases, since by (i) $\xi_1(z)$ decreases with d. Choose ρ small to make the second integral small. Then choose d small to make $|d\zeta_1(1-\rho, 0)/dz - 1|$ small; this makes the first term small since the integral converges uniformly as $d \to 0$: at $-\infty$ since $f_1(x)$ is exponentially small, and at -1 since $\zeta_1(z) + 1 = O|z+1|^{1+\epsilon/\pi}$, and $f_1[1 - \xi_1(z)]$ has only a logarithmic singularity. Thus $I_2 \to 0$ as $d \to 0$.

To estimate the integral

$$\int_1^{\infty} |f(\xi - \xi')| \left| \frac{dz}{d\zeta'} - 1 \right| d\zeta'$$

map $0 > \eta > -h$ on D_1, keeping B fixed. We find $\delta \to 0$ as $d \to 0$.

It remains to prove the assumptions:

(i) follows by an argument essentially similar to that of the Lemma of Sec. 3.

(ii) follows from Th. I(i) of Sec. 3 where it is shown that

$$\left| 1 - \frac{d\zeta_1}{dz} \right| < \frac{2d}{h} \frac{e^{\pi(x+1)/h}}{(1 - e^{\pi(x+1)/h})^2}.$$

(iii) follows from the fact that the boundary of the domain of $\log d\zeta_1/dz$

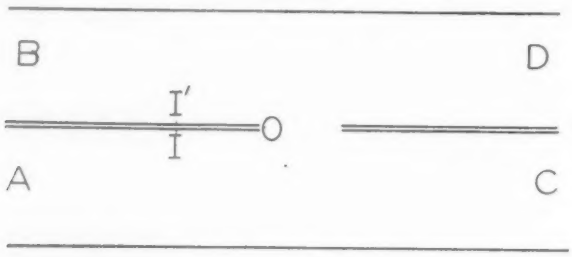

FIG. 6

is simple (since the trapezoid is convex) so that $\log (d\zeta_1/dz)$ and $d\zeta_1/dz$ are schlicht in D_1 and $d^2\zeta_1/dz^2 \neq 0$. It follows that $\partial^2\zeta/\partial x^2 \neq 0$ on $z < -1$, i.e. $\partial^2\zeta/\partial x^2 < 0$. Since $d\zeta/dz = \partial\zeta/\partial x$ on $z < -1$, $d\zeta/dz$ decreases monotonically in $-\infty < z < -1$.

THE DIFFRACTION OF A PLANE WAVE THROUGH A GRATING*

BY

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Summary. The problem of diffraction and scattering of a normally incident plane wave of sound by an infinite plane grating consisting of infinitely thin, coplanar, equally spaced (b apart) strips with parallel edges is solved. The potentials on the two sides of the screen are written as Fourier expansions in terms of the velocity in the aperture, and an integral equation for this velocity is determined. An impedance parameter Z whose real part is the transmission coefficient, is defined, and it is shown that the real and imaginary parts of the reciprocal of this parameter may both be specified by variational expressions, which are absolute minima for the solution to the aforementioned integral equation. An alternative formulation, in terms of the pressure discontinuity across the screen, is given, leading to an integral equation and to variational expressions for the real and imaginary parts of $(1 - Z)^{-1}$. A solution to the integral equation is given which reduces the problem to the solution of an infinite number of simultaneous equations. It is shown that solving only one of these equations gives a solution which is essentially a solution to Laplace's equation, while if N equations are solved the terms neglected are of the order $N^{-1}[(1 - 4b^2/N^2\lambda^2)^{-1/2} - 1]$ or less, where λ is the wave length. The solution is extended to the case of a vertically polarized electromagnetic incident wave by direct analogy and to the case of a horizontally polarized electromagnetic wave by a transformation which is a special case of Babinet's principle. The integral equations for an aperture of finite thickness are set up, and an approximate solution including only first order terms in thickness/wavelength is given. Results are given in the form of curves for the transmission coefficient vs. aperture opening for several ratios of grating spacing to wave length less than unity. For the special case of a half open grating the transmission coefficient is plotted out to that value of spacing/wavelength at which the results agree with Kirchhoff's theory.

Introduction. The problem to be solved is the scattering and diffraction of a plane wave of sound which is (essentially) normally incident upon a plane grating consisting of an infinite number of infinitely thin parallel ribbons of width $b - d$ whose centers are spaced a distance b apart, as shown in Fig. 1. Since, by symmetry, the normal velocity must vanish at the planes which form the perpendicular bisectors of the ribbons, shown by dashed lines in Fig. 1, the problem is the same as the calculation of the transmission of sound through a symmetrical window of opening $b - d$ placed perpendicular to two, infinite, parallel plates, a distance b apart. This problem, subject to certain restrictions on the frequency, has been treated by the present author.[1] For the sake of completeness and in order to investigate further the limitations on the frequency, the analysis will be carried out independently of earlier papers.

The solution will be formulated in terms of the normal velocity in the plane of the grating, which, by virtue of Huygen's principle, may be regarded as a source distribution.

*Received March 18, 1948.

[1] John W. Miles, *The analysis of plane discontinuities in cylindrical tubes*, J. Acoust. Soc. Am. **17**, Part I: 259-272, Part II: 272-285 (1946). The variational methods used therein were due to J. S. Schwinger.

The potentials on the two sides of the screen may then be regarded as composed of the solution corresponding to no aperture in the screen (i.e., complete reflection) plus radiation from the unknown source distribution over the aperture. The normal derivatives of the potentials on the two sides of the screen will then automatically satisfy appropriate boundary conditions, while the requirement that the potentials themselves be continuous

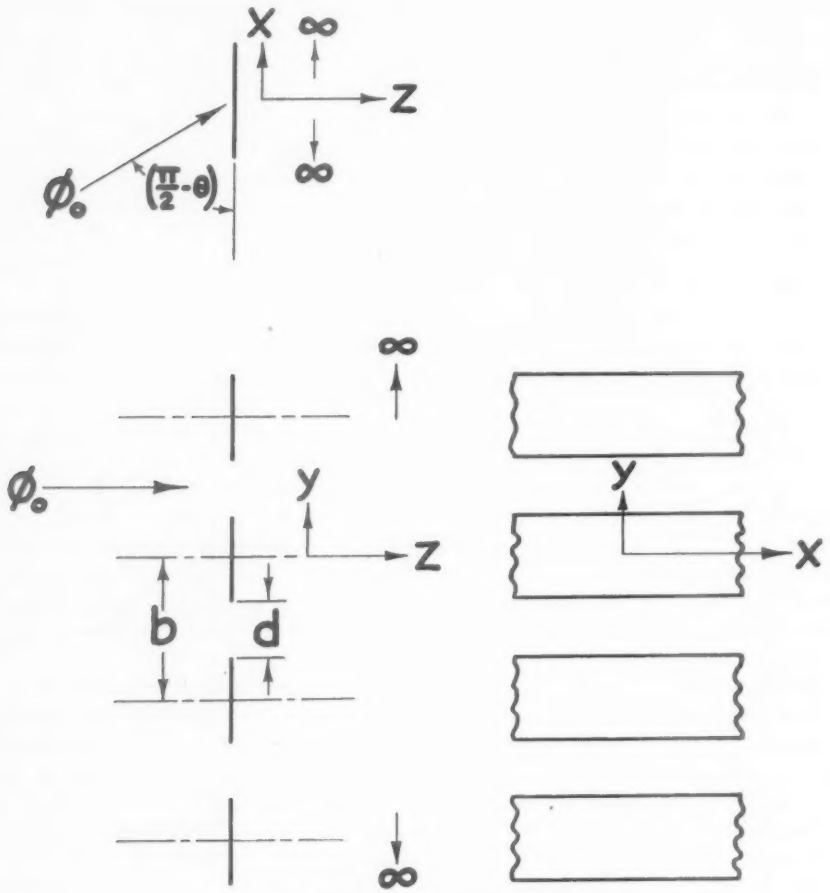

FIG. 1. Three views of infinite grating.

across the aperture yields an integral equation for the determination of the source distribution (or aperture velocity). An alternative formulation in terms of the pressure discontinuity across the screen will also be given.

In many cases, it is sufficient to know the transmission coefficient for the grating, i.e., the ratio of the energy density transmitted through the grating to that in the incident wave. It is convenient to introduce a complex impedance parameter, whose real part is the transmission coefficient, and this leads to a variational formulation of the problem.

Velocity potentials. Is is known[2] that if the velocity vector $\mathbf{q}$ is specified as the gradient of a potential ϕ and a harmonic time variation $\exp(j\omega t)$ is assumed, the potential satisfies the scalar Helmholtz equation

$$\nabla^2\phi(x, y, z) + k^2\phi(x, y, z) = 0, \qquad (1) \qquad k = \omega/c, \tag{2}$$

$$\mathbf{q}(x, y, z, t) = \exp(j\omega t)\nabla\phi(x, y, z), \tag{3}$$

while the pressure is given by

$$p(x, y, z, t) = -j\omega\rho_0 \exp(j\omega t)\phi(x, y, z). \tag{4}$$

If the incident wave from $z = -\infty$ has its wave normal in the xz plane, it may be designated by

$$\phi_{inc}(x, y, z) = \Phi_0 \exp[-jk(x \sin\theta + z \cos\theta)], \tag{5}$$

where the axes are as shown in Fig. 1. It is evident (since there is no discontinuity in the x direction) that the dependence on the x coordinate, together with the amplitude Φ_0, may be removed by writing

$$\phi(x, y, z) = \Phi_0 \exp(-jkx \sin\theta)\varphi(y, z), \tag{6}$$

$$\nabla^2\varphi(y, z) + \kappa^2\varphi(y, z) = 0, \tag{7}$$

where

$$\kappa = k \cos\theta. \tag{8}$$

It follows that the solution for $\theta \neq 0$ may be obtained from the solution for $\theta = 0$ simply by replacing k in the latter by $k \cos\theta$ and multiplying the result by $\exp(-jkx \sin\theta)$. Accordingly, only the case $\theta = 0$ will be discussed in the following sections.

A solution to the foregoing equations which is periodic in y, corresponding to the periodicity of the grating, which corresponds to the incident wave (5), and which satisfies the boundary conditions

$$\frac{\partial\varphi}{\partial y}(0, z) = \frac{\partial\varphi}{\partial y}(b, z) = 0, \tag{9a}$$

$$\frac{\partial\varphi}{\partial z}(y, 0) = 0 \qquad , \qquad y \text{ in } \tau, \tag{9b}$$

$$= -jkf(y), \qquad y \text{ in } \sigma, \tag{9c}$$

is given by

$$z < 0 : \varphi_1(y, z) = 2 \cos kz - k \sum_{n=0}^{\infty} b_n\kappa_n^{-1} \exp(j\kappa_n z) \cos(n\pi y/b), \tag{10a}$$

$$z > 0 : \varphi_2(y, z) = \qquad k \sum_{n=0}^{\infty} b_n\kappa_n^{-1} \exp(-j\kappa_n z) \cos(n\pi y/b), \tag{10b}$$

[2] P. M. Morse, *Vibration and sound*, McGraw-Hill, New York, 1936.

$$b_n = \left(\frac{2 - \delta_n^0}{b}\right) \int_\sigma f(\eta) \cos(n\pi\eta/b)\, d\eta, \tag{11}$$

$$\kappa_n = (k^2 - n^2\pi^2/b^2)^{1/2} \qquad \text{if} \qquad k > (n\pi/b), \tag{12a}$$

$$= -j(n^2\pi^2/b^2 - k^2)^{1/2} \qquad \text{if} \qquad k < (n\pi/b). \tag{12b}$$

The aperture region is denoted by σ, while the obstacle is denoted by τ. The leading term $2 \cos kz$ in φ_1 represents the solution corresponding to total reflection and would be the complete solution (i.e., $b_n \equiv 0$) if the grating were replaced by an unperforated screen. The remaining terms represent the waves which are radiated due to the presence of the aperture σ, their Fourier amplitudes being chosen to correspond to a normal velocity $-jkf(y)$, as specified by Eq. (9c), in the aperture σ. The vanishing of this latter velocity on the obstacle τ, as required by Eq. (9b), is ensured by integrating, cf. Eq. (11), only over σ. The boundary conditions of Eq. (9a) specify the periodicity of the solution and are satisfied by the choice of the expansion functions $\cos(n\pi y/b)$. The behavior of the propagation constants κ_n at their branch points, cf. Eq. (12), is chosen to ensure attenuation of the non-propagated waves.

The only remaining boundary condition to be satisfied is that of continuity of pressure across the aperture, i.e.,

$$\varphi_1(y, 0) = \varphi_2(y, 0), \qquad y \text{ in } \sigma. \tag{13}$$

Substituting Eqs. (10) and (11) in Eq. (13) yields the integral equation

$$\int_\sigma K(y, \eta) f(\eta)\, d\eta = 1, \qquad y \text{ in } \sigma, \tag{14}$$

$$K(y, \eta) = K(\eta, y) = \sum_{n=0}^{\infty} \left(\frac{2 - \delta_n^0}{b}\right) Y_n \cos(n\pi y/b) \cos(n\pi\eta/b), \tag{15}$$

$$Y_n = (k/\kappa_n) = [1 - (n\lambda/2b)^2]^{-1/2}, \qquad (2b/\lambda) > n, \tag{16a}$$

$$= j[(n\lambda/2b)^2 - 1]^{-1/2}, \qquad (2b/\lambda) < n. \tag{16b}$$

Before discussing the solution of Eq. (14) it will be convenient to discuss alternative formulations of the problem and to introduce an impedance parameter.

Aperture impedance. The energy flow per unit x width through the grating is given by

$$P = \frac{1}{2} Rl \int_\sigma \bar{p}(x, y, 0, t) q_z(x, y, 0, t)\, dy. \tag{17}$$

Substituting p from Eq. (4), q_z from Eq. (9c) and observing from Eqs. (10) and (13) that $\varphi = 1$ in the aperture, Eq. (17) reduces to

$$P = \frac{1}{2} \rho_0 c k^3 \Phi_0^2 \cos\theta \left[Rl \int_\sigma f(y)\, dy \right]. \tag{18}$$

The aperture impedance will therefore be defined by

$$Z = R + jX = \frac{1}{b} \int_\sigma f(y)\, dy = b_0 \tag{19}$$

where R and X are the real (resistive) and imaginary (reactive) parts of Z.

Now in Kirchhoff's theory[3] of diffraction it is assumed that the fields in the aperture are the same as they would be in the absence of τ, i.e., $\phi_1(x, y, 0) = \phi_2(x, y, 0) = \phi_{inc}(x, y, 0)$, in which case $f(y) = 1$. The ratio of actual tramsmitted energy to that predicted by Kirchhoff's theory is therefore given by

$$P/P_K = \left(\frac{d}{b}\right)^{-1}\left[\frac{Rl}{b}\int_\sigma f(y)\,dy\right] = \left(\frac{b}{d}\right)R. \tag{20}$$

The reactance term X is a measure of the non-propagated energy stored in the vicinity of the aperture, i.e. of the standing waves.

It may be remarked that the impedance defined by Eq. (19) is closely analogous in the quantity utilized by Rayleigh in his studies of apertures in cavities.[4] On the other hand, the impedance differs from that defined in the paper quoted in Footnote 1 in including the "radiation resistance" R, and, as will be shown subsequently, also in the definition of the imaginary component.

Variational formulation. Multiplying both sides of Eq. (14) by $f(y)\,dy$, integrating over the aperture, and dividing through by Eq. (19) yields

$$Z^{-1} = Y = G + jB = b\left[\int_\sigma \bar{f}(y)\,dy\int_\sigma f(\eta)\,d\eta\right]^{-1}\int_\sigma \bar{f}(y)\,dy\int_\sigma f(\eta)K(y,\eta)\,d\eta. \tag{21}$$

Equation (21) is a variational equation of a type first studied by Schwinger[5], although in the present case both the kernel K and the unknown function f are complex. In order to further study the properties of Eq. (21), it is expedient to reduce it to two real expressions. Thus, write

$$f(y) = f_1(y) + jf_2(y), \tag{22}$$

$$K(y,\eta) = K_1(y,\eta) + jK_2(y,\eta) \tag{23}$$

where f_1 and f_2 are both real functions, and K_1 and K_2 are the real and imaginary parts of the kernel, corresponding to the Y_n of Eqs. (16a,b).

Substituting Eqs. (22) and (23) in Eq. (21), remembering the symmetry of the kernel in (y, η), yields

$$G = b\left[\left(\int_\sigma f_1\,dy\right)^2 + \left(\int_\sigma f\,dy\right)^2\right]^{-1} \cdot\left[\int_\sigma f_1\,dy\int_\sigma f_1K_1\,d\eta + \int_\sigma f_2\,dy\int_\sigma f_2K_1\,d\eta\right], \tag{24a}$$

$$B = b\left[\left(\int_\sigma f_1\,dy\right)^2 + \left(\int_\sigma f_2\,dy\right)^2\right]^{-1} \cdot\left[\int_\sigma f_1\,dy\int_\sigma f_1K_2\,d\eta + \int_\sigma f_2\,dy\int_\sigma f_2K_2\,d\eta\right]. \tag{24b}$$

[3] J. A. Stratton, *Electromagnetic theory*, McGraw-Hill, New York, 1941. Since the Kirchhoff calculation is generally carried out by integrating the flux through a hemisphere, the results designated as Kirchhoff herein might more precisely be labelled geometric.

[4] Lord Rayleigh, *Theory of sound*, Dover Press, New York, 1946.

[5] At the M. I. T. Radiation Laboratory (1941-1945); see also the paper quoted in Footnote 1.

Similarly the integral equation (14) may be rewritten:

$$\int_\sigma [K_1(y, \eta)f_1(\eta) - K_2(y, \eta)f_2(\eta)]\, d\eta = 1, \tag{25a}$$

$$\int_\sigma [K_1(y, \eta)f_2(\eta) + K_2(y, \eta)f_1(\eta)]\, d\eta = 0. \tag{25b}$$

Now by varying G and B as given by Eqs. (24), with respect to both f_1 and f_2, it is found that the functions f_1 and f_2 which satisfy Eqs. (25) make both G and B stationary. Moreover, from the symmetry of K_1 and K_2, both G and B are positive definite forms (since they will be sums of squares), and it follows that they are absolute minima with respect to variations about the true functions f_1 and f_2. The formal proof is analogous to that given in Appendix C of the paper quoted in Footnote 1 for the variational expression obtained by assuming that f_1 and f_2 differ only by a constant in Eq. (24b).

A closer analogy to the cases treated in the author's earlier paper[1] is obtained by assuming $\lambda > b$, as assumed therein, in which case it follows from Eq. (15) that $K_1 = 1$, and Eq. (24a) reduces to

$$G = 1 \qquad \text{if} \qquad \lambda > b \cos\theta \tag{26}$$

In this case, it may be assumed that $f_2 = 0$, and it is found that the susceptance B is half that for the symmetrical window mentioned in the Introduction to the present paper.

Alternative formulation. The problem can also be formulated in terms of the pressure discontinuity across the screen, defined by

$$\gamma(y) = \frac{1}{2}[\varphi_1(y, 0^-) - \varphi_2(y, 0^+)]. \tag{27}$$

In terms of $\gamma(y)$, the potentials may be written

$$z \lesseqgtr 0 : \varphi_2^1(y, z) = \exp(-jkz) \pm \sum_{n=0}^{\infty} d_n \exp(\pm j\kappa_n z) \cos(n\pi y/b), \tag{28}$$

$$d_n = \left(\frac{2 - \delta_n^0}{b}\right) \int_\tau \gamma(\eta) \cos(n\pi\eta/b)\, d\eta. \tag{29}$$

The potentials given by Eq. (28) evidently satisfy Eq. (13) implicitly, since the Fourier coefficients are so defined. Moreover, differentiation shows that the normal velocity is continuous at $z = 0$, so that Eq. (9c) is satisfied. It remains to satisfy Eq. (9b) by demanding the integral equation.

$$\int_\tau H(y, \eta)\gamma(\eta)\, d\eta = 1, \qquad y \text{ in } \tau, \tag{30}$$

$$H(y, \eta) = \sum_{n=0}^{\infty} \left(\frac{2 - \delta_n^0}{b}\right) Z_n \cos(n\pi y/b) \cos(n\pi\eta/b), \tag{31}$$

$$Z_n = Y_n^{-1} = k^{-1}[k^2 - (n\pi/b)^2]^{1/2}. \tag{32}$$

From Eqs. (9b) and (27) it is evident that $\gamma(y)$ and $f(y)$ are related by

$$f(y) = 1 - \int_\tau H(y, \eta)\gamma(\eta)\, d\eta, \tag{33a}$$

$$\gamma(y) = 1 - \int_\sigma K(y, \eta)f(\eta)\, d\eta, \tag{33b}$$

so that the Fourier coefficients of f and γ are related by

$$b_n + d_n = \delta_n^0 . \tag{34}$$

For $n = 0$, Eqs. (34) and (19) yield

$$Z = 1 - \frac{1}{b}\int_\tau \gamma(\eta)\, d\eta. \tag{35}$$

Multiplying both sides of Eq. (30) by $\bar{\gamma}(y)\, dy$, integrating over the obstacle, and dividing through by $1 - Z$ from Eq. (35) yields the variational expression

$$(1 - Z)^{-1} = b\left[\int_\tau \bar{\gamma}(y)\, dy \int_\tau \gamma(\eta)\, d\eta\right]^{-1}\left[\int_\tau \bar{\gamma}(y)\, dy \int_\tau \gamma(\eta)H(y, \eta)\, d\eta\right]. \tag{36}$$

Now Eq. (36) is exactly analogous to Eq. (21), and it may be shown that its real and imaginary parts are absolute minima with respect to variations of $\gamma(y)$ about the true value of $\gamma(y)$ satisfying the integral equation (30). In order to compare the results so obtained, it may be observed that

$$(1 - Z)^{-1} = [(1 - G)^2 + B^2]^{-1}[(1 - G) + jB]. \tag{37}$$

For the special case where Eq. (26) holds, it is seen that Eq. (36) makes $1/B$ a minimum, and the results of Eqs. (24b) and (36) therefore bound B from above and below. This case was treated before.[1] For the case where the wavelength is less than b Eq. (24) bounds G and B from above, while Eq. (36) bounds $[(1 - G) + (1 - G)^{-1}B^2]$ and $[B + B^{-1}(1 - G)^2]$ from below.

Solution to the integral equation. In the problem at hand, the most appropriate attack appears to be an approximate solution to the integral equation (14). The accuracy of the solution can then be checked by substituting the solution so obtained in the variational expression (21) and by calculating γ from Eqs. (33) and (34) and substituting in Eq. (36).

In order to solve Eq. (14), the first step is to rewrite the kernel $K(y, \eta)$ as

$$K(y, \eta) = b^{-1} + j(2k/\pi)\sum_{n=1}^{\infty} n^{-1}\cos(n\pi y/b)\cos(n\pi\eta/b) + (2k/\pi)\sum_{n=1}^{\infty}\Delta_n \cos(n\pi y/b)\cos(n\pi\eta/b), \tag{38}$$

$$\Delta_n = n^{-1}\{[(2b/n\lambda)^2 - 1]^{-1/2} - j\}, \qquad (2b/\lambda) > n, \tag{39a}$$

$$= jn^{-1}\{[1 - (2b/n\lambda)^2]^{-1/2} - 1\}, \qquad (2b/\lambda) < n. \tag{39b}$$

The second term in Eq. (32) may be written in closed form, since

$$2\sum_{n=1}^{\infty} n^{-1}\cos(nz)\cos(n\zeta) = -\ln[2\,|\cos z - \cos\zeta\,|], \tag{40}$$

whence

$$\begin{aligned} K(y,\eta) = b^{-1} &- j(k/\pi)\ln[2\,|\cos(\pi y/b) - \cos(\pi\eta/b)\,|] \\ &+ (2k/\pi)\sum_{n=1}^{\infty}\Delta_n\cos(n\pi y/b)\cos(n\pi\eta/b). \end{aligned} \tag{41}$$

Before substituting Eq. (41) in Eq. (14), it is expedient to introduce the change of variable

$$\cos(\pi y/b) = \alpha\cos\theta, \qquad \text{(42a)} \qquad \alpha = \sin(\pi d/2b), \tag{42b}$$

$$u(\theta) = \alpha(1-\alpha^2\cos^2\theta)^{-1/2} f[(b/\pi)\cos^{-1}(\alpha\cos\theta)]\sin\theta. \tag{43}$$

Substituting Eqs. (42) and (43) in Eq. (14) yields

$$\frac{1}{\pi}\int_0^{\pi} bK(\theta,\psi)u(\psi)\,d\psi = 1, \qquad 0 < \theta < \pi, \tag{44}$$

$$\begin{aligned} bK(\theta,\psi) &= 1 - j(kb/\pi)\ln[2\alpha\,|\cos\theta - \cos\psi\,|] \\ &\quad + (2kb/\pi)\sum_{n=1}^{\infty}\Delta_n\cos(n\pi y/b)\cos(n\pi\eta/b) \\ &= [1 - j(kb/\pi)\ln\alpha] + j(2kb/\pi)\sum_{n=1}^{\infty} n^{-1}\cos n\theta\cos n\psi \\ &\quad + (2kb/\pi)\sum_{n=1}^{\infty}\Delta_n\cos(n\pi y/b)\cos(n\pi\eta/b), \end{aligned} \tag{45}$$

where y, η and θ, ψ in the last term are understood to be related by Eq. (42). Now expand $u(\theta)$ in the Fourier series

$$u(\theta) = \sum_{n=0}^{\infty} a_n\cos(n\theta). \tag{46}$$

Substituting Eq. (46) in Eq. (44) yields

$$\begin{aligned} &[1 - j(kb/\pi)\ln\alpha]a_0 + j(kb/\pi)\sum_{n=1}^{\infty} a_n n^{-1}\cos(n\theta) \\ &\quad + (kb/\pi)\sum_{r=1}^{\infty}\Delta_r\cos(r\pi y/b)\sum_{s=0}^{\infty} a_s I_{rs} = 1, \end{aligned} \tag{47}$$

$$I_{rs} = \frac{2}{\pi}\int_0^{\pi}\cos[r\cos^{-1}(\alpha\cos\psi)]\cos(s\psi)\,d\psi. \tag{48}$$

Moreover, since I_{mn} are the Fourier expansion coefficients of $\cos(n\pi y/b)$, Eq. (47) may be rewritten

$$[1 - j(kb/\pi)\ln\alpha]a_0 + j(kb/\pi)\sum_{n=1}^{\infty} a_n n^{-1}\cos(n\theta) + (kb/\pi)\sum_{r=1}^{\infty}\Delta_r\sum_{n=0}^{\infty}(1+\delta_n^0)^{-1}I_{rn}\cos(n\theta)\sum_{s=0}^{\infty}a_s I_{rs} = 1. \tag{49}$$

Now, since Eq. (49) must be valid for all θ between 0 and π, it follows that

$$\delta_0^n[1 - j(kb/\pi)\ln\alpha]a_0 + (kb/\pi)\{j(1-\delta_n^0)n^{-1}a_n + \sum_{r=1}^{\infty}(1+\delta_n^0)^{-1}\Delta_r I_{rn}\sum_{s=0}^{\infty}I_{rs}a_s\} = \delta_n^0, \qquad n = 0, 1, 2, \cdots \tag{50}$$

which is an infinite set of simultaneous equations for the determination of the d_n.

While it would have been simpler to solve the integral equation (12) by a direct Fourier expansion in $\cos(n\pi y/b)$, leading to a set of simultaneous equations for the Fourier coefficients, the transformation to the θ coordinate allows the separation of the $\ln\alpha$ term, which is essentially the susceptance in the limit of zero frequency.

The Irs integrals may be easily evaluated by an identity due to Jacobi[6]

$$\int_0^{\pi}\phi(\cos\theta)\cos(n\theta)\,d\theta = [1\cdot 3\cdot\cdots(2n-1)]^{-1}\int_0^{\pi}\frac{d^n\phi(\cos\theta)}{d(\cos\theta)^n}(\sin\theta)^{2n}\,d\theta. \tag{51}$$

Now

$$\cos(r\pi\eta/b) = \sum_{p=0}^{r} c_p^r\alpha^p(\cos\theta)^p, \tag{52a}$$

$$c_p^r = \text{coeff. of } (\cos x)^p \text{ in expansion of } \cos(nx), \tag{52b}$$

whence it follows that

$$\frac{d^s\cos(r\pi\eta/b)}{d(\cos\theta)^s} = \sum_{p=s}^{r} p(p-1)\cdots(p-s+1)c_p^r\alpha^p(\cos\theta)^{p-s}, \qquad s \le r$$
$$= 0, \qquad r < s \tag{53}$$

Using the result

$$\int_0^{\pi}(\cos\theta)^m(\sin\theta)^n\,d\theta = B\left(\frac{m+1}{2}, \frac{n+1}{2}\right) \tag{54}$$

where B is the Beta function, it follows that

$$I_{rs} = \frac{2}{\pi}[1\cdot 3\cdots(2s-1)]^{-1}\sum_{p=s}^{r}\cdot p(p-1)\cdots(p-s+1)c_p^r\alpha^p B\left(\frac{p-s+1}{2}, \frac{2s+1}{2}\right), \qquad r \ge s, \tag{55}$$
$$= 0, \qquad r < s.$$

[6] H. Bateman, *Partial differential equations*, Dover Press, New York, 1944, p. 463. (Note that the first integral given by Bateman on the top of p. 464 is incorrect and is not used here.)

In the present case, due to symmetry, only even values of r and s need be considered, so that $p - s$ and $2s$ will be even, in which case

$$\frac{2}{\pi} B\left(\frac{p-s+1}{2}, \frac{2s+1}{2}\right) = \left[2^{p+s-1}\left(\frac{p-s}{2}\right)!\left(\frac{p+s}{2}\right)!s!\right]^{-1}(p-s)!(2s)! \tag{56}$$

In an approximate solution, only a finite number, say N, of Eqs. (50) will be solved. The error in solving N equations will evidently be of order Δ_N at worst. While the Δ_n (cf. Eq. 39) generally decrease with increasing n, each possesses a singularity at $b/\lambda = n/2$, so that it will generally be necessary to satisfy the inequality

$$N > (2b/\lambda). \tag{57}$$

It should also be remembered that, due to symmetry, the odd A_n vanish identically, and the number of equations to be solved for a given N is actually only $(N + 1)/2$.

Electromagnetic problem. The case of a plane electromagnetic wave normally incident on a grating (assumed perfectly conducting at radio frequencies or perfectly reflecting at optical frequencies) is analogous to the above acoustical problem since, due to the normal incidence, it is possible to derive all six components of the electromagnetic field from a single component of the incident field, provided that the cases of vertical (electric field) and horizontal polarization of the incident wave are treated separately.

Vertically polarized wave. Consider first the vertically polarized incident wave consisting of the fields

$$\mathbf{H}_0 = \quad \mathbf{i}H_x^0 \exp(-jkz), \tag{58a}$$

$$\mathbf{E}_0 = -\mathbf{j}H_x^0 \exp(-jkz), \tag{58b}$$

which satisfy Maxwell's equations (MKS units)

$$\nabla \times \mathbf{E} = -jk\mathbf{H}, \tag{59a}$$

$$\nabla \times \mathbf{H} = jk\mathbf{E}, \tag{59b}$$

$$\nabla \cdot \mathbf{E} = 0, \tag{60a}$$

$$\nabla \cdot \mathbf{H} = 0. \tag{60b}$$

The boundary conditions to be satisfied in the screen are

$$E_x = E_y = H_z = 0, \tag{61}$$

only two of which are independent, according to Eqs. (59). Finally, each of the field components satisfies the scalar Helmholtz equation (1), which follows directly from Maxwell's equations (59) and (60).

Now in the case of the incident wave (58), it is evident that the entire problem may be formulated in terms of field components H_x, E_y, and E_z. Moreover, from Eqs. (59), it is seen that

$$jkE_y(y, z) = \frac{\partial H_x}{\partial z}(y, z), \tag{62a}$$

$$jkE_z(y, z) = -\frac{\partial H_x}{\partial y}(y, z), \tag{62b}$$

which satisfy Eqs. (59) and (60). Therefore, it is evident from comparing Eqs. (5) and (58a), (9b) and (61b), and (9c) and (61b) that

$$H_x^0 \sim \Phi_0 \,, \tag{63a}$$

$$H_x(y, z) \sim \Phi_0 \varphi(y, z), \tag{63b}$$

$$E_y(y, 0) \sim \Phi_0 f(y). \tag{63c}$$

It follows that the foregoing acoustical problem yields a direct solution to the electromagnetic problem when the incident wave is plane and vertically polarized. (It may also be shown that the pressure discontinuity γ is analogous to the current induced in the screen by the magnetic field.)

Horizontally polarized wave. A horizontally polarized incident field consists of the components

$$\mathbf{E}_0 = \mathbf{i} E_x^0 \exp(-jkz), \tag{64a}$$

$$\mathbf{H}_0 = \mathbf{j} E_x^0 \exp(-jkz). \tag{64b}$$

Consider now the solution

$$z \lesseqgtr 0 : E_x^{1,2}(y, z) = E_x^0 \left\{ \exp(-jkz) - k \sum_{n=0}^{\infty} b_n \kappa_n^{-1} \exp(\pm j\kappa_n z) \cos(n\pi y/b) \right\}, \tag{65a}$$

$$z \lesseqgtr 0 : H_y^{1,2}(y, z) = E_x^0 \left\{ \exp(-jkz) \pm \sum_{n=0}^{\infty} b_n \exp(\pm j\kappa_n z) \cos(n\pi y/b) \right\}. \tag{65b}$$

It is evident that Eq. (65) corresponds to Eq. (64) and satisfies Eqs. (59) and (60). Moreover, if $f(y)$ is a solution to Eq. (14) in the plane $z = 0$, the following conditions are satisfied:

$$E_x^1(y, 0) = E_x^2(y, 0) \quad , \quad 0 \leq y \leq b, \tag{66a}$$

$$E_x^1(y, 0) = E_x^2(y, 0) = 0 \quad , \quad y \text{ in } \sigma, \tag{66b}$$

$$H_y^1(y, 0) = H_y^2(y, 0) = E_x^0 \,, \quad y \text{ in } \tau, \tag{66c}$$

$$H_y^1(y, 0^-) - H_y^2(y, 0^+) = 2E_x^0 f(y) \quad , \quad y \text{ in } \sigma. \tag{66d}$$

Hence, it follows that Eq. (65) represents a solution to Maxwell's equations which satisfies boundary conditions appropriate to the complementary grating, i.e., that grating obtained by interchanging aperture σ and obstacle τ of the original grating. This result is a special case of Babinet's principle, which, in its rigorous form, has been discussed by Copson[7], following Booker, by Watson[8], and by Schwinger[9].

From Eq. (66d) it is evident that $f(y)$ in the present case represents the current flowing in the screen, whereas in the case of the vertically polarized incident wave $f(y)$ represented the tangential electric field in the aperture. Conversely, it may be observed that an alternative solution for the present case is given by

[7]E. T. Copson, *An integral equation method of solving plane diffraction problems.* Proc. Roy. Soc. London (A) **186**, 100-118 (1946).

[8]W. H. Watson, *Physical principles of waveguide transmission*, Clarendon Press, Oxford, 1947.

[9]J. Schwinger, M. I. T. Radiation Lab. Reports, Cambridge, Mass., 1941-1945.

$$z < 0 : E_x^1(y, z) = E_x^0\left\{-2j \sin(kz) + \sum_{n=0}^{\infty} d_n \exp(j\kappa_n z) \cos(n\pi y/b)\right\}, \tag{67a}$$

$$z > 0 : E_x^2(y, z) = E_x^0 \sum_{n=0}^{\infty} d_n \exp(-j\kappa_n z) \cos(n\pi y/b), \tag{67b}$$

$$z < 0 : H_y^1(y, z) = E_x^0\left\{2 \cos(kz) - k^{-1} \sum_{n=0}^{\infty} d_n \kappa_n \exp(j\kappa_n z) \cos(n\pi y/b)\right\}, \tag{67c}$$

$$z > 0 : H_y^2(y, z) = E_x^0 k^{-1} \sum_{n=0}^{\infty} d_n \kappa_n \exp(-j\kappa_n z) \cos(n\pi y/b). \tag{67d}$$

The boundary conditions (66a) and (66b) are automatically satisfied by Eqs. (67a,b), since

$$E_x^1(y, 0) = E_x^2(y, 0) = E_x^0 \gamma(y) \tag{68}$$

The boundary condition (66c) is satisfied if $\gamma(y)$ is a solution to the integral equation (30). Hence, the aperture field in the horizontally polarized problem is equivalent to the screen current in the complementary problem (vertically polarized incident wave and complementary grating). This result is another aspect of Babinet's principle.

Transmission and reflection coefficients. If the transmission coefficient is defined as the ratio of the energy flow to the right of the screen to the energy flow in the undisturbed incident wave, it follows from Eqs. (18) and (19) that

$$T = Rl(Z) = Rl(a_0) \tag{69}$$

for the acoustic problem.

In the electromagnetic problem the transmission coefficient is given by integrating Poynting's vector over the aperture and dividing by the energy flow (between $y = 0, b$) in the incident wave to obtain

$$T_v = b^{-1}(H_z^0)^{-2} Rl \int_\sigma \mathbf{E}(y, 0) \times \overline{\mathbf{H}}(y, 0)\, dy, \tag{70a}$$

$$= b^{-1} Rl \int_\sigma f(y)\, dy = Rl(Z) = Rl(a_0) \tag{70b}$$

for the vertically polarized case. For the case of horizontal polarization

$$T_h = b^{-1}(E_z^0)^{-2} Rl \int_\tau \mathbf{E}(y, 0) \times \overline{\mathbf{H}}(y, 0)\, dy, \tag{71a}$$

$$= b^{-1} Rl \int_\tau \gamma(y)\, dy, \tag{71b}$$

$$= Rl\left[1 - b^{-1} \int_\sigma f(y)\, dy\right], \tag{71c}$$

$$= 1 - T_v\,. \tag{71d}$$

The relation $T_h + T_v = 1$ where T_h and T_v are the transmission coefficients for complementary gratings and complementary polarization, is predicted by Babinet's principle.[8]

From conservation of energy, it is evident that the reflection coefficients are given by $1 - T$. In the case where $\lambda > b$, only a plane wave will be reflected, and there will be a simple standing wave pattern to the left of the screen. In this case, as may be seen from Eq. (10), it is convenient to define the complex reflection coefficient $1 - b_0$, which will then give the phase of the standing waves.

Thick grating. Experimental measurements (electromagnetic) in wave guides[10] have shown that the principal errors in theoretically calculated diffractions through thin windows, etc. are due to the assumption of zero thickness and that a thickness of as little as 0.005 wavelength may cause a deviation of several per cent between theory and experiment, although the experimental results approach those predicted by theory as thickness is decreased. Lord Rayleigh[4] predicted that thickness would also be quite important in the acoustical problem, although he does not offer any direct reason or cite any supporting experimental evidence. Moreover, the deviations from an ideal fluid in acoustical experiments are such as to make precision measurements more difficult than in analogous electromagnetic problems.

Consider the window of finite thickness t and symmetrically located opening d between two parallel plates a distance b apart, as shown in Fig. 2. The faces of the window will be

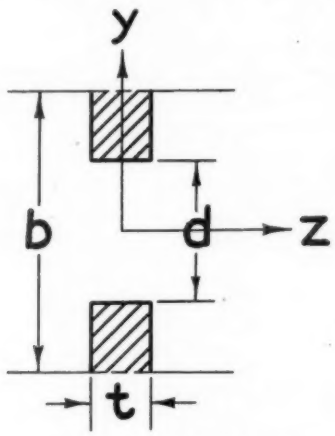

FIG. 2. Section of thick grating.

assumed to lie in the planes $z = \pm t/2$. The potentials in the three z regions, corresponding to the incident wave (5) are given by

$$z < -t/2 : \varphi_1(y, z) = 2 \exp(jkt/2) \cos[k(z + t/2)] - k \sum_{n=0}^{\infty} b_n^1 \kappa_n^{-1} \exp[j\kappa_n(z + t/2)] \cos(n\pi y/b), \tag{72a}$$

[10] *Waveguide handbook*. M. I. T. Radiation Laboratory Report, 41—1/23/45 to be published by McGraw-Hill, New York.

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$$z < 0 : H_y^1(y, z) = E_x^0\left\{2 \cos(kz) - k^{-1} \sum_{n=0}^{\infty} d_n \kappa_n \exp(j\kappa_n z) \cos(n\pi y/b)\right\}, \tag{67c}$$

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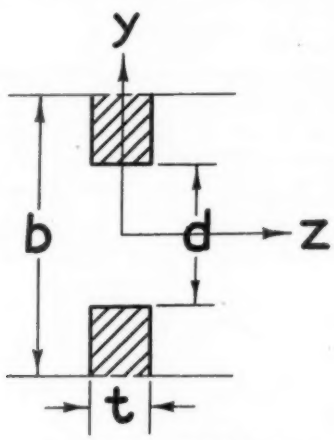

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[10] *Waveguide handbook*. M. I. T. Radiation Laboratory Report, 41—1/23/45 to be published by McGraw-Hill, New York.

$$-t/2 < z < +t/2 : \varphi_{12}\, y, z) = -jk \sum_{n=0}^{\infty} \left(\frac{2 - \delta_n^0}{d}\right)\zeta_n^{-1}\Big\{(\sin [\zeta_n(z + t/2)]$$

$$+ \cot (\zeta_n t) \cos [\zeta_n(z + t/2)]) \int_\sigma f_1(\eta)$$

$$\times \cos\left[(n\pi/d)\left(\eta - \frac{b - d}{2}\right)\right] d\eta$$

$$+ (\sin [\zeta_n(z - t/2)] - \cot (\zeta_n t) \cos [\zeta_n(z - t/2)]) \tag{72b}$$

$$\times \int_\sigma f_2(\eta) \cos\left[(n\pi/d)\left(\eta - \frac{b - d}{2}\right)\right] d\eta\Big\}$$

$$\times \cos\left[(n\pi/d)\left(y - \frac{b - d}{2}\right)\right],$$

$$z > +t/2 : \varphi_2(y, z) = k \sum_{n=0}^{\infty} b_n^2\, \kappa_n^{-1} \exp [-j\kappa_n(z - t/2)] \cos (n\pi y/b), \tag{72c}$$

$$\zeta_n = [k^2 - (n\pi/d)^2]^{1/2}. \tag{72d}$$

The $b_n^{1,2}$ are given by substituting $f_{1,2}$ in Eq. (11).

The expansion functions and coefficients in Eq. (72) have been chosen so as to satisfy the boundary condition that the normal velocity vanish at all walls, and reduce to the normal velocities in the two apertures, $-jkf_1(y)$ and $-jkf_2(y)$ at $z = \mp t/2$ respectively. These velocities are then determined by the requirement that φ_1, φ_{12}, and φ_2 be continuous across the two apertures, which yields the simultaneous integral equations

$$\int_\sigma [K_{11}(y, \eta)f_1(\eta) + K_{12}(y, \eta)f_2(\eta)]\, d\eta = \exp (-jkt/2) \qquad y \text{ in } \sigma \tag{73a}$$

$$\int_\sigma [K_{21}(y, \eta)f_1(\eta) + K_{22}(y, \eta)f_2(\eta)]\, d\eta = 0, \tag{73b}$$

$$K_{11}(y, \eta) = K_{22}(y, \eta) = \frac{1}{2} \sum_{n=0}^{\infty} \left\{\left(\frac{2 - \delta_n^0}{b}\right)(k/\kappa_n) \cos (n\pi y/b) \cos (n\pi\eta/b)\right.$$

$$-j\left(\frac{2 - \delta_n^0}{d}\right)(k/\zeta_n) \cot (\zeta_n t) \cos\left[(n\pi/d)\left(y - \frac{b - d}{2}\right)\right] \tag{74a}$$

$$\left.\times \cos\left[(n\pi/d)\left(\eta - \frac{b - d}{2}\right)\right]\right\},$$

$$K_{12}(y, \eta) = K_{21}(y, \eta) = \left(\frac{j}{2}\right) \sum_{n=0}^{\infty} \left(\frac{2 - \delta_n^0}{d}\right)(k/\zeta_n) \csc (\zeta_n t)$$

$$\times \cos\left[(n\pi/d)\left(y - \frac{b - d}{2}\right)\right] \cos\left[(n\pi/d)\left(\eta - \frac{b - d}{2}\right)\right]. \tag{74b}$$

The integral equations (73) can be solved systematically by expanding f_1 and f_2 in the functions $\cos[(n\pi/d)\{\eta - (b-d)/2\}]$, multiplying the integral equations by $\cos[(m\pi/d)\cdot\{y - (b-d)/2\}]\,dy$, and integrating over the aperture to obtain a doubly infinite set of simultaneous coefficients for the determination of the expansion coefficients. However, only a simple approximate solution will be given herein, since the effect contemplated is presumably small.

It will be assumed that the fields $f_{1,2}$ are similar to the solution for $t = 0$ but differ in phase by an amount proportional to t/λ, viz:

$$f_{1,2}(y) = [1 + j\epsilon_{1,2}kt + \delta_{1,2}(kt)^2]f_0(y) + 0[(t/\lambda)^3] \tag{75}$$

where f_0 is presumed to be a solution of the integral equation (14). The terms in $\delta_{1,2}$ will be dropped in the final results, but it is necessary to include them in the intermediate steps. It will also be assumed that $\epsilon_{1,2}$ are constants, although this is not strictly correct (even up to terms of order t/λ), since the nature of the singularity at the edge of the aperture is quite sensitive to thickness.

The kernel functions of Eq. (74) may be rewritten

$$K_{11}(y, \eta) = \frac{1}{2}K_0(y, \eta) - j(2kt)^{-1}K'_{12}(y, \eta) + O[(t/\lambda)^3], \tag{76a}$$

$$K_{12}(y, \eta) = j(2\kappa t)^{-1}K'_{12}(y, \eta) + O[(t/\lambda)^3], \tag{76b}$$

$$K'_{12}(y, \eta) = \sum_{n=0}^{\infty}\left(\frac{2 - \delta_n^0}{d}\right)(k/\zeta_n)^2 \cos\left[(n\pi/d)\left(y - \frac{b-d}{2}\right)\right] \times \cos\left[(n\pi/d)\left(\eta - \frac{b-d}{2}\right)\right] \tag{76c}$$

where the terms of order t/λ can be shown to vanish identically. K_0 is the kernel of Eq. (14).

Substituting Eqs. (75) and (76) in Eqs. (73), and recalling that f_0 satisfies Eq. (14), Eqs. (73) reduce to

$$\frac{1}{2}(1 + jkt\epsilon_1) + \frac{1}{2}(\epsilon_1 - \epsilon_2)\int_\sigma K'_{12}f_0\,d\eta - \frac{1}{2}jkt(\delta_1 - \delta_2)\int_\sigma K'_{12}f_0\,d\eta + O[(t/\lambda)^3] = 1 + \frac{1}{2}jkt, \tag{77a}$$

$$\frac{1}{2}(1 + jkt\epsilon_2) - \frac{1}{2}(\epsilon_1 - \epsilon_2)\int_\sigma K'_{12}f_0\,d\eta + \frac{1}{2}jkt(\delta_1 - \delta_2)\int_\sigma K'_{12}f_0\,d\eta + O[(t/\lambda)^3] = 0. \tag{77b}$$

Equating powers of t in Eq. (77a) yields

$$(\epsilon_1 - \epsilon_2)\int_\sigma K'_{12}f_0\,d\eta = 1, \tag{78}$$

while adding Eqs. (77a) and (77b) yields

$$\epsilon_1 + \epsilon_2 = 1. \tag{79}$$

Multiplying Eq. (78) by dy and integrating over the aperture σ yields (cf. Eq. 19):

$$(\epsilon_1 - \epsilon_2)d^{-1} \int_\sigma f_0 \, d\eta = (\epsilon_1 - \epsilon_2)(b/d)Z = 1. \tag{80}$$

Solving Eqs. (79) and (80) yields

$$\epsilon_{1,2} = \frac{1}{2}[1 \pm (d/b)Y] \tag{81}$$

where Y is the admittance computed for zero thickness.

The net change in the transmission coefficient is evidently given by

$$\begin{aligned} T(t)/T(0) &= |\, 1 + jkt\epsilon_2 \,|^2 \\ &= |\, 1 + j(kt/2) - j(kt/2)(d/b)(G + jB) \,|^2 \\ &= 1 + (d/b)(kt)B + O[(t/\lambda)^2] \end{aligned} \tag{82}$$

where B is the susceptance calculated at $t = 0$.

Actually, the correction given by the foregoing approximate analysis is generally too small. This is probably due to the fact that the nature of the singularity near the edge of the aperture is extremely sensitive to thickness, whereas the fields in Eq. (75) possess the same singularity as in the case of the infinitely thin aperture. In order to take this factor into account, one might solve the static problem $k = 0$ by conformal mapping and establish a thickness correction based on the difference between the field so obtained and its limiting form at zero thickness. Unfortunately, the results obtained are rather too complex to be useful.

Uniqueness of solution. Bouwkamp,[11] in reviewing a recent paper by Copson,[7] has pointed out that solutions to Maxwell's equations such as Copson's,[7] and also Bethe's,[12] are not unique inasmuch as they neglect the possibility of a free charge on the edge of the aperture, with its resulting scalar potential. The present analysis is open to the same criticism. Unfortunately, a rigorous discussion is complicated by the singularity at the edge of the aperture. Moreover, the results predicted by Bethe's theory[12] have been verified[13] with a precision which appears to settle the question from a practical point of view, at least insofar as the distant fields are concerned. It must be remarked, however, that the charge distribution suggested by Bouwkamp might well affect the distribution of the field in the aperture without markedly affecting the field at a distance, since the latter depends primarily on the mean value of the former and not on its distribution. Andrews[14] has recently made some precision measurements on the electromagnetic field in a circular aperture which might prove to be a valuable confirmation of the theoretically

[11] C. J. Bouwkamp, Math. Revs. 8, 180 (1947).

[12] H. A. Bethe, *Theory of diffraction by small holes*, Phys. Rev. **66**, 163-182 (1944).

[13] *Loc. cit.*, [10], and other M. I. T. Radiation Laboratory reports of groups 41 and 43.

[14] C. L. Andrews, *Diffraction pattern of a circular aperture at short distances*, Phys. Rev. **71**, 777-781 (1947).

calculated field in the aperture. Unfortunately, existing solutions for this problem[12,15] are not sufficiently accurate (for holes whose dimensions are comparable to the wavelength) to warrant comparison. Moreover, the fields in the immediate vicinity of the aperture will almost certainly be quite sensitive to thickness.

Numerical results. The I_{rs} computed from Eqs. (55) and (56) for use up to $r = s = 6$ are given by

r/s	0	2	4	6
0	2	0	0	0
2	$[2(\alpha^2 - 1)]$	α^2	0	0
4	$[2(3\alpha^2 - 1)(\alpha^2 - 1)]$	$[4\alpha^2(\alpha^2 - 1)]$	α^4	0
6	$[2(10\alpha^4 - 8\alpha^2 + 1)(\alpha^2 - 1)]$	$[3\alpha^2(5\alpha^2 - 3)(\alpha^2 - 1)]$	$[6\alpha^4(\alpha^2 - 1)]$	α^6.

Choosing $N = 6$ in Eq. (5), a_0, a_2, and a_4 are given by

$$a_0^{-1} = 1 + 2j(b/\lambda) \ln(\alpha^{-1}) + 4(b/\lambda)(1 - \alpha^2)^2 D^{-1}\{\Delta_2 + (3\alpha^2 - 1)^2\Delta_4 - 6j\alpha^4(\alpha^4 - 2\alpha^2 + 3)\Delta_2\Delta_4\}, \tag{84a}$$

$$D = 1 - 2j\alpha^4[\Delta_2 + 2(9\alpha^4 - 16\alpha^2 + 8)\Delta_4] - 8\alpha^{12}\Delta_2\Delta_4, \tag{84b}$$

$$(a_2/a_0) = 4\alpha^2(1 - \alpha^2)D\{j[\Delta_2 - 4(1 - \alpha^2)(3\alpha^2 - 1)\Delta_4] + 4\alpha^8\Delta_2\Delta_4\}, \tag{84c}$$

$$(a_4/a_0) = 8\alpha^4(1 - \alpha^2)D\Delta_4[j(3\alpha^2 - 1) + 4\alpha^6\Delta_2] \qquad (b/\lambda) < 3 \tag{84d}$$

For $\lambda \gg b$, the simplest approximation is given by setting $\Delta_2 = \Delta_4 = 0$ in Eq. (84a) to obtain*

$$Rl(a_0) = \{1 + 4(b/\lambda)^2 \ln^2[\csc(\pi d/2b)]\}^{-1} \qquad (b/\lambda) \ll 1 \tag{85}$$

A more accurate expression is given by retaining Δ_2, as given by Eq. (39b) to obtain

$$Rl(a_0) = \{1 + 4(b/\lambda)^2(\ln[\csc(\pi d/2b)] + \Gamma)^2\}^{-1}, \qquad (b/\lambda) < 1, \tag{86a}$$

$$\Gamma = \{1 + ([1 - (b/\lambda)^2]^{-1/2} - 1)\sin^4(\pi d/2b)\}^{-1} \times \{[1 - (b/\lambda)^2]^{-1/2} - 1\}\cos^4(\pi d/2b). \tag{86b}$$

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[15]Lord Rayleigh, *On the passage of waves through apertures in plane screens and allied problems*, Philos. Mag. **43**, 259-272 (1897) and *Sci. Papers* **4**, 283-296; also, *On the incidence of aerial and electric waves upon small obstacles in the form of ellipsoids or electric cylinders, and on the passage of electric waves through a circular aperture in a conducting screen*, Philos. Mag. **44**, 28-52 (1897) and *Sci. Papers* **4**, 305-327.

*Since this paper was written the author has found that the result given in equation 85 was first found by Horace Lamb, *On the diffraction in transmission of electric waves by a metallic grating*, Proc. London Math. Society **29**, 523-544 (1898). Moreover a bibliography on problems related to this, such as diffraction through a grating of wires, etc. is given by H. Bateman, *Electrical and optical wave motion*, Cambridge University Press, 1915, p. 89.

For $(b/\lambda) > 1$, the expression for $Rl(a_0)$ became rather too complex to be worth writing out explicitly, and it is simpler to calculate a_0 directly from Eq. (84a).

The result given in Eq. (86) for $b < \lambda$ of course reduces to Eq. (85) for small values of b/λ. Now, where b/λ was less than unity, the accuracy of the susceptance, as given by Eq. (24b) was investigated[1] by using the variational formulations of Eqs. (29b) and (36), and the susceptance corresponding to the result given by Eq. (86a) was found to be in error by about 0.1% for $d/b = 1/2$ and $b/\lambda = 1/2$, while the susceptance corresponding to the approximation of Eq. (85) was in error by 10% for these figures. For $d/b = 1/2$ and $b/\lambda = 1$ the approximation corresponding to Eq. (86a) is in error about 1%, while that corresponding to Eq. (85) is in error by over 74%.

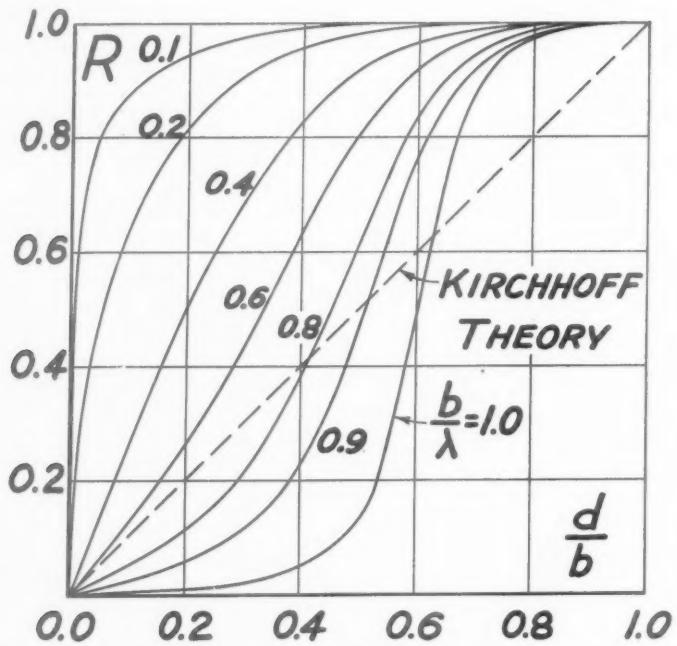

FIG. 3. Resistive component of grating impedance (equivalent to transmission coefficient).

In Figs. 3 and 4 R and X are plotted vs. the aperture ratio d/b for $b/\lambda = 0$, 0.1, 0.2, 0.4, 0.6, 0.8, 0.9, and 1.0, using, for the most part, data previously calculated[10]. In Fig. 5, R is plotted vs b/λ from 0 to 1.9, for $d/b = 1/2$, the calculations having been made for $N = 4$.

The result obtained by substituting $f(y) = \text{const.}$ in Eq. (21) is

$$G + jB = 1 + 2 \sum_{n=1}^{\infty} [1 - (n\lambda/b)^2]^{-1/2} (n\pi d/b)^{-2} \sin^2 (n\pi d/b) \tag{87}$$

and is also plotted in Fig. 5 for values of b/λ up to 3.8. The accuracy of the transmission factor corresponding to Eq. (87) may be estimated by comparing with the results ob-

tained from the solution (84) which are accurate to better than 1% (estimated) for $b/\lambda < 2$. The accuracy of Eq. (87) will, however, improve with increasing values of b/λ.

Moreover, for large values of b/λ, the imaginary part of Eq. (87) may be neglected, while the real part is given approximately by

$$G = 1 + 2\sum_{n=1}^{\infty} (n\pi d/b)^{-2} \sin^2 (n\pi d/b) = 1 + (b/\pi d)^2 S(\pi d/b), \tag{88}$$

$$S(x) = \sum_{n=1}^{\infty} n^{-2} \sin^2 (nx). \tag{89}$$

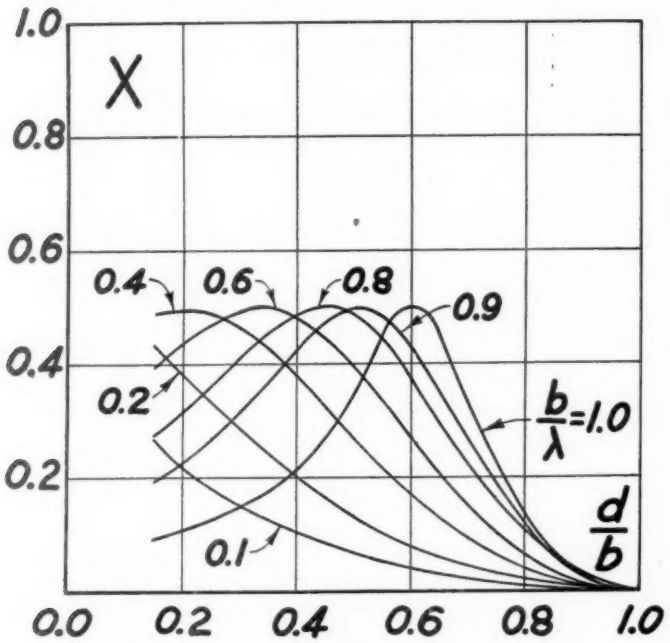

FIG. 4. Reactive component of grating impedance.

Differentiating Eq. (89) with respect to x yields

$$\frac{dS(x)}{dx} = \sum_{n=1}^{\infty} n^{-1} \sin (2nx) = Im \sum_{n=1}^{\infty} n^{-1} \exp (j2nx)$$
$$= Im [\ln (1 - e^{j2x})^{-1}] = \left(\frac{\pi}{2} - x\right). \tag{90}$$

Integrating Eq. (90) with respect to x, and requiring $S(0) = 0$, yields

$$S(x) = \frac{x}{2}(\pi - x). \tag{91}$$

Substituting Eq. (91) in Eq. (88) yields

$$G = \frac{b}{d}. \tag{92}$$

Since B is negligible

$$T = \frac{1}{G} = \frac{d}{b}. \tag{93}$$

in agreement with Kirchhoff's theory, so that Eq. (87) is certainly correct for sufficiently large values of b/λ. Eq. (93) is borne out by Fig. 5.

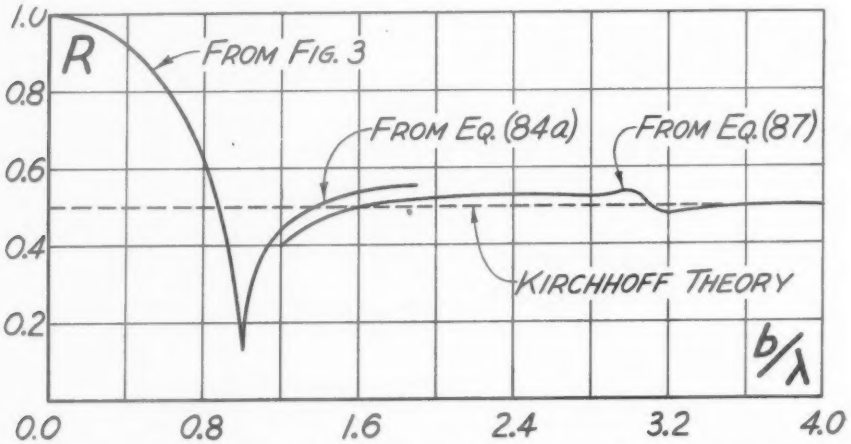

FIG. 5. Transmission coefficient for half open grating as computed from Eqs. (84a), (87), and Kirchhoff theory.

The value $T = d/b$ predicted by Kirchhoff's theory is plotted in Figs. 3 and 5 for comparison, and it is seen that the agreement is not good, even for values of b/λ approaching 2, but is evidently satisfactory for $b/\lambda > 4$. Morse and Rubenstein,[16] in calculating the exact results for the diffraction of a normally incident plane wave on a single slit, obtained agreement for values of d/λ greater than about unity. On the other hand, Morse and Rubenstein's results show that it is necessary to go to quite large values of b/λ in the case of oblique incidence before agreement with Kirchhoff's results is obtained. Unfortunately, the present method cannot be applied to the problem of oblique incidence for the grating, except perhaps in those special cases where the periodicity of the incident wave in a plane of constant z is the same as the periodicity of the grating.

It is also of interest to note the sharp resonance at $b/\lambda = 1$, in contrast to the slight dip at $b/\lambda = 3$. There are no resonances at $b/\lambda = 2$ or 4 due to the additional symmetry introduced at $d/b = 1/2$.

[16] P. M. Morse and P. V. Rubenstein, *The diffraction of waves by ribbons and slits*, Phys. Rev. **54**, 895-898 (1938).

SUPERSONIC FLOW OVER BODIES OF REVOLUTION*

BY

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1. Introduction. The purpose of this paper is to present a method of determination of the flow of a compressible fluid about the head of a pointed body of revolution. This method is based on the assumption that the actual flow may be approximated by a perturbation of the conical flow about the tip of the body, an assumption that will hold for contours of small curvature. The flow is furthermore supposed to be steady and axially symmetric; viscosity, heat conduction and external forces are considered to be negligible. Since the entropy perturbation is of the same order as the velocity perturbation, it must be taken into account.

2. Differential equations of the flow. Let us suppose that the velocity $\mathbf{q}$ of the flow and the local speed a of sound are measured in units of the speed c of efflux into a vacuum, that density ρ, pressure p, and absolute temperature T are measured in units of their respective stagnation values ρ_0 , p_0 , T_0 in the free stream, and that the entropy S is measured in units of c_v ; further we put $S_0 = 0$. Then we have, in view of our basic assumptions,[1]

Bernoulli's equation:

$$1 - q^2 = \frac{2}{\gamma - 1} a^2 = T, \tag{1}$$

isentropy along streamlines:

$$p/\rho^\gamma = g(S) = e^S, \tag{2}$$

the equation of continuity:

$$\nabla \cdot (\mathbf{q} T^{1/(\gamma-1)}) = 0, \tag{3}$$

and the vorticity equation:

$$(\nabla \times \mathbf{q}) \times \mathbf{q} = \frac{a^2}{\gamma(\gamma - 1)} \nabla S. \tag{4}$$

Introducing spherical coordinates r, θ, φ, so that the tip of the body lies at the origin $r = 0$, and its axis coincides with the line $\theta = 0$, and denoting the velocity components in the direction of increasing r, θ, φ by X, Y, Z, respectively, (see Fig. 1), we have for axially symmetric flow, since $Z = 0$, $\partial/\partial\varphi = 0$,

$$\nabla \cdot (\mathbf{q} T^{1/(\gamma-1)}) = \frac{\partial}{\partial r}(X T^{1/(\gamma-1)}) + \frac{2}{r}(X T^{1/(\gamma-1)}) + \frac{1}{r}\frac{\partial}{\partial \theta}(Y T^{1/(\gamma-1)}) + \frac{\cot\theta}{r}(Y T^{1/(\gamma-1)}) = 0.$$

*Received April 28, 1948.

[1]See, eg., R. Sauer, *Theoretische Einführung in die Gasdynamik*, Springer, Berlin, 1943, p. 132.

Now

$$\frac{\partial}{\partial r} T^{1/(\gamma-1)} = -\frac{XX_r + YY_r}{a^2} T^{1/(\gamma-1)}, \qquad X_r \equiv \partial X/\partial r, \qquad \text{etc.},$$

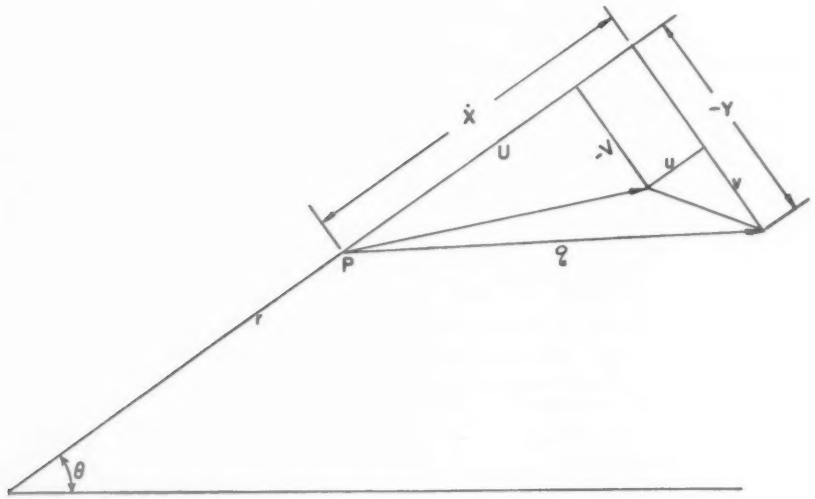

FIG. 1.

so that (3) becomes, after multiplication by $ra^2T^{-1/(\gamma-1)}$,

$$rX_r(a^2 - X^2) - XY(X_\theta + rY_r) + Y_\theta(a^2 - Y^2) + a^2(2X + Y\cot\theta) = 0. \tag{5}$$

The components of $\nabla \times \mathbf{q}$ in the r, θ, φ directions, respectively, are 0, 0, $(1/r)(X_\theta - rY_r - Y)$. Therefore,

$$(\nabla \times \mathbf{q}) \times \mathbf{q} = -\frac{1}{r}(X_\theta - rY_r - Y)(Y\mathbf{k}_1 - X\mathbf{k}_2),$$

where $\mathbf{k}_1$ and $\mathbf{k}_2$ are unit vectors in the r and θ directions. By Eq. (4) then,

$$X_\theta - rY_r - Y = -\frac{a^2}{\gamma(\gamma-1)} \cdot \frac{rS_r}{Y} = \frac{a^2}{\gamma(\gamma-1)} \frac{S_\theta}{X}. \tag{6}$$

In case of a conical axially symmetric field, $X = U(\theta)$ and $Y = V(\theta)$ are functions of θ only, and the flow is necessarily irrotational, so that Eqs. (5) and (6) become[2]

$$-UV^2 + (a_0^2 - V^2)V' + a_0^2(2U + V\cot\theta) = 0, \tag{7}$$

$$U' - V = 0, \tag{8}$$

with $a_0^2 = (1/2)(\gamma-1)(1 - U^2 - V^2)$, $U' \equiv dU/d\theta$.

It will in the following be assumed that U, V are the velocity components in the

[2]These equations are identical with those first derived by Taylor and Maccoll, Proc. Roy. Soc. (A) 139, 278 (1939).

conical field of the nose of the body of revolution, and that they have already been determined as functions of θ. We now assume that

$$X = U + u, \qquad Y = V + v, \tag{9}$$

and that the velocity perturbations u, v are small compared with U, V, making second order terms such as u^2, uu_r, etc., negligible. Under these conditions Eqs. (5) and (6) become, after some simplifications,

$$ArU_r + B(u_\theta + rv_r) + Cv_\theta + Du + Ev = 0, \tag{10}$$

$$u_\theta - rv_r + Fu + Gv + H = 0, \tag{11}$$

with

$$A = a_0^2 - U^2, \qquad B = -UV, \qquad C = a_0^2 - V^2,$$

$$D = 2a_0^2 - V^2 - (\gamma - 1)U(2U + V\cot\theta) - (\gamma - 1)UV',$$

$$E = a_0^2\cot\theta - (\gamma - 1)V(2U + V\cot\theta) - (\gamma + 1)VV',$$

$$F = \frac{S_\theta}{\gamma}\left(\frac{a_0^2}{(\gamma - 1)U^2} + 1\right), \qquad G = \frac{S_\theta}{\gamma}\frac{V}{U} - 1, \qquad H = -\frac{S_\theta}{\gamma}\frac{a_0^2}{(\gamma - 1)U}.$$

3. System of characteristic differential equations. Since $u = u(r, \theta)$, $v = v(r, \theta)$, whence $du = u_r\,dr + u_\theta\,d\theta$, $dv = v_r\,dr + v_\theta\,d\theta$, the characteristic equations[3] of the system (10) and (11) may be obtained from the condition that the matrix

$$\begin{vmatrix} Ar & B & Br & C & Du + Ev \\ 0 & 1 & -r & 0 & Fu + Gv + H \\ dr & d\theta & 0 & 0 & -du \\ 0 & 0 & dr & d\theta & -dv \end{vmatrix}$$

has the rank 3.

The vanishing of the determinant of the first four columns leads to

$$Ar^2\,d\theta^2 - 2Br\,dr\,d\theta + Cdr^2 = 0,$$

whence[4]

$$dr - \frac{B \pm R}{C}\,r\,d\theta = 0, \tag{12}$$

with

$$R = (B^2 - AC)^{1/2} = [a_0^2(q_0^2 - a_0^2)]^{1/2}, \qquad q_0^2 = U^2 + V^2.$$

Putting

$$m(\theta) = (B + R)/C, \qquad n(\theta) = (B - R)/C,$$

[3]For the theory of characteristics see R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, J. Springer, Berlin, 1937, vol. 2, ch. V.

[4]Note that for supersonic flow always $C > 0$.

we obtain for the equation of the first[5] family η = const. of characteristics

$$dr - rm\, d\theta = 0, \tag{13}$$

and for the second[5] family ξ = const.

$$dr - rn\, d\theta = 0. \tag{14}$$

It is easy to show that Eqs. (13) and (14) are also the equations for the characteristics of the conical field, as follows. Since (see Fig. 2)

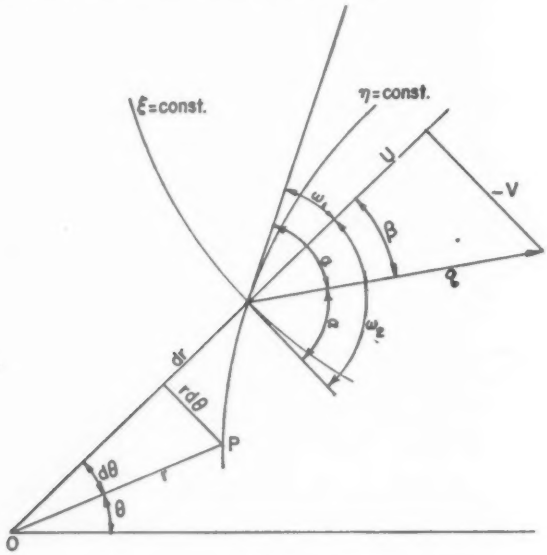

FIG. 2.

$$\omega_1 = \alpha - \beta, \qquad \omega_2 = -\alpha - \beta,$$

and $\tan \alpha = a_0(q_0^2 - a_0^2)^{-1/2}$, $\tan \omega_1 = rd\theta/dr$, $\tan \beta = -V/U$, we get

$$\pm \frac{r\, d\theta}{dr} = \frac{a_0^2 - V^2}{\mp UV + a_0(q_0^2 - a_0^2)^{1/2}} = \frac{C}{\pm B + R}$$

along a characteristic of the first and second family, respectively. These relationships, however, are identical with Eqs. (13) and (14).

Integration of these equations leads to

$$r/r_1 = \exp \int_{\theta_1}^{\theta} m(\theta)\, d\theta,$$

find

$$r/r_2 = \exp \int_{\theta_2}^{\theta} n(\theta)\, d\theta,$$

[5]Looking in the direction of **q**, the "first" family is the family running forward to the left of **q**, the "second" family is the one running to the right of **q**.

fi 1, 2, P are three points located as shown in Fig. 3. These relationships may also be expressed in the form

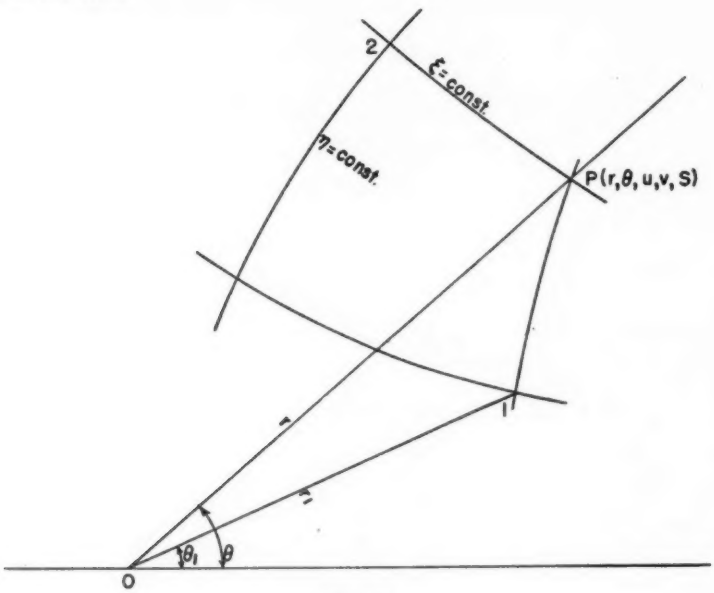

FIG. 3.

$$r/r_1 = M(\theta)/M(\theta_1), \tag{15}$$

$$r/r_2 = N(\theta)/N(\theta_2), \tag{16}$$

where $$M(\theta) = \exp \int_{\theta_s}^{\theta} m(\theta)\, d\theta, \qquad N(\theta) = \exp \int_{\theta_s}^{\theta} n(\theta)\, d\theta.$$

For later use we note here also that

$$r_2/r_1 = (r_2/r)(r/r_1) = \exp\left(\int_{\theta}^{\theta_2} n d\theta + \int_{\theta_1}^{\theta} m d\theta\right),$$

whence

$$M(\theta)/N(\theta) = (r_2/r_1)M(\theta_1)/N(\theta_2). \tag{17}$$

To determine the velocity perturbations u, v at P we make use of the fact that also necessarily

$$\begin{vmatrix} Ar & B & C & Du + Ev \\ 0 & 1 & 0 & Fu + Gv + H \\ dr & d\theta & 0 & -du \\ 0 & 0 & d\theta & -dv \end{vmatrix} = 0.$$

In view of $mn = A/C$, $m + n = 2B/C$ this results in

$$dv + ndu + Kd\theta = 0 \text{ along } \eta = \text{const.}, \tag{18}$$

$$dv + mdu + Ld\theta = 0 \text{ along } \xi = \text{const.}, \tag{19}$$

with
$$K = C^{-1}[(D - RF)u + (E - RG)v - RH],$$

$$L = C^{-1}[(D + RF)u + (E + RG)v + RH].$$

Finally, since $dS = S_r dr + S_\theta d\theta$, and, by Eq. (6), $-rXS_r = YS_\theta$, we have

$$dS = S_\theta d\theta(1 - mY/X) \qquad \text{along } \eta = \text{const.}, \tag{20}$$

and
$$dS = S_\theta d\theta(1 - nY/X) \qquad \text{along } \xi = \text{const.} \tag{21}$$

It is thus seen that in order to compute K, L, and dS it is necessary to know S_θ. This quantity may be found as follows. Since

$$S = S[\xi(r, \theta), \eta(r, \theta)], \text{ we have } S_\theta = S_\xi \xi_\theta + S_\eta \eta_\theta .$$

But

$$\begin{pmatrix} \xi_r & \xi_\theta \\ \eta_r & \eta_\theta \end{pmatrix} = \frac{1}{J} \begin{pmatrix} \theta_\eta & -r_\eta \\ -\theta_\xi & r_\xi \end{pmatrix}$$

with $J = \partial(r, \theta)/\partial(\xi, \eta)$. By Eqs. (13), (14) $r_\xi = rm\theta_\xi$, $r_\eta = rn\theta_\eta$, whence

$$J = (m - n)r\theta_\xi\theta_\eta \qquad \text{and} \qquad S_\theta = \frac{1}{m - n}\left(m\frac{S_\eta}{\theta_\eta} - n\frac{S_\xi}{\theta_\xi}\right). \tag{22}$$

If, then, the quantities θ and S are known at the points 1, 2, and P of a characteristic net (see Fig. 3), then approximately

$$S_\theta = \frac{1}{m - n}\left(m\frac{S - S_2}{\theta - \theta_2} - n\frac{S - S_1}{\theta - \theta_1}\right). \tag{23}$$

4. Boundary conditions. *A. At the body.* Let the equation of the contour of the body of revolution in a meridian plane be

$$r = f(\theta). \tag{24}$$

Since the contour must be a streamline, we have

$$f'(\theta) = rX/Y.$$

Introducing the perturbations u, v defined in Eqs. (9), we obtain

$$ru - f'(\theta)v + rU - f'(\theta)V = 0. \tag{25}$$

B. At the shock wave. Here the laws of conservation of mass, of momentum, and of energy must be satisfied, and, moreover, the change in entropy must be calculated.

From the three conservation laws it is easy to deduce first the continuity of the tangential velocity component: $q_t = q_{1t}$. (Fig. 4) Along the shock, then,

$$\frac{dr}{d\theta} = r\frac{q_1 \sin\ \theta + Y}{q_1 \cos\ \theta - X}, \tag{26}$$

where $\mathfrak{q}_1$ is the free stream velocity. This relationship may also be expressed in the form

$$\tan\ \theta_w = \frac{q_1 - X \cos\ \theta + Y \sin\ \theta}{X \sin\ \theta + Y \cos\ \theta}. \tag{27}$$

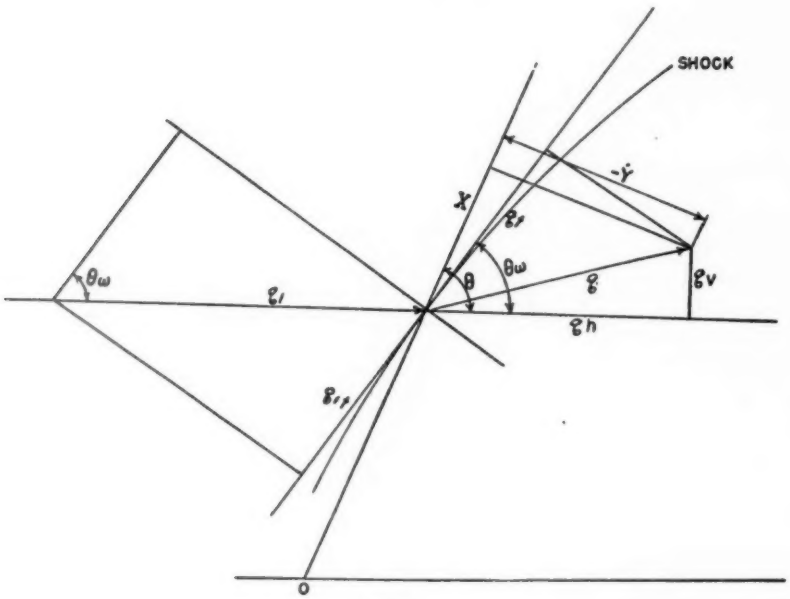

FIG. 4.

The velocity perturbations u, v along the shock wave may be found as follows. Let q_h and q_v be the horizontal and vertical components of $\mathfrak{q}$; then (see Fig. 4)

$$q_h = X \cos\ \theta - Y \sin\ \theta, \qquad q_v = X \sin\ \theta + Y \cos\ \theta. \tag{28}$$

Along the shock wave q_h and q_v satisfy the equation of the shock polar

$$(q_1 - q_h)^2(q_h - b) - q_v^2(e - q_h) = 0 \tag{29}$$

with

$$b = \mu/q_1\ , \qquad e = \mu/q_1 + (1 - \mu)q_1\ , \qquad \mu = (\gamma - 1)/(\gamma + 1).$$

Making use of Eqs. (28) and introducing the perturbations u, v we find that Eq. (29) leads to

$$\Gamma u + \Delta v + \Lambda = 0, \tag{30}$$

where

$$\Gamma = (3U^2 + V^2)\cos\theta - 2UV\sin\theta - 2U[(2q_1 + b)\cos^2\theta + e\sin^2\theta] + q_1(1+\mu)V\sin 2\theta + (2\mu + q_1^2)\cos\theta,$$

$$\Delta = -(U^2 + 3V^2)\sin\theta + 2UV\cos\theta - 2V[(2q_1 + b)\sin^2\theta + e\cos^2\theta] + q_1(1+\mu)U\sin 2\theta - (2\mu + q_1^2)\sin\theta,$$

$$\Lambda = U^3\cos\theta - V^3\sin\theta - UV(U\sin\theta - V\cos\theta) - U^2[(2q_1 + b)\cos^2\theta + e\sin^2\theta] - V^2[(2q_1 + b)\sin^2\theta + e\cos^2\theta] + q_1(1+\mu)UV\sin 2\theta + (2\mu + q_1^2)(U\cos\theta - V\sin\theta) - \mu q_1 .$$

To determine the entropy variation along the shock we notice first that

$$\frac{dS}{d\theta} = S_r\frac{dr}{d\theta} + S_\theta = S_\theta\left(1 - \frac{Y}{X}\frac{1}{r}\frac{dr}{d\theta}\right). \tag{31}$$

Now from

$$S = \log(p/\rho^\gamma) = \log(p/p_1) + \gamma\log(\rho_1/\rho) + S_1 , \tag{32}$$

and the Rankine-Hugoniot equations

$$\frac{p}{p_1} = \frac{2\gamma M_1^2\sin^2\theta_w - (\gamma - 1)}{\gamma + 1},$$

$$\frac{\rho_1}{\rho} = \frac{(\gamma - 1)\sin^2\theta_w + 2/M_1^2}{(\gamma + 1)\sin^2\theta_w}$$

it may be shown that

$$\frac{dS}{d\theta} = \frac{2\gamma ks}{t^2}\cdot\frac{(\tan^2\theta_w - t)^2}{\tan\theta_w(\tan^2\theta_w - k)(\tan^2\theta_w + s)}\cdot\frac{d\theta_w}{d\theta} \tag{33}$$

with $t^{-1} = M_1^2 - 1$, $k^{-1} = [2\gamma/(\gamma - 1)]M_1^2 - 1$, $s^{-1} = [(\gamma - 1)/2]M_1^2 + 1$.

Once θ_w is known, Eq. (33) permits the calculation of $dS/d\theta$, and then Eq. (31) that of S_θ. This will be described in more detail in the following section.

5. Computational procedure. Let the value of $M_1 > 1$ and the contour of the body of revolution be given (see Fig. 5). We replace the nose 0′10″ of the body by a small tangential cone 010″, and compute the conical flow U, V determined by M_1, θ_s, thus obtaining a number of starting points 2, 3, 4, $\cdots$ on the initial η-characteristic 11′. We use these points to get a characteristic net as shown in Fig. 5. Each point of this net belongs to one of these four types:

a. It lies on the initial η − characteristic 11′, e.g. 2;
b. It lies on the contour, e.g. 10;
c. It lies in the "interior", e.g. 5′;
d. It lies on the shock wave, e.g. 6′.

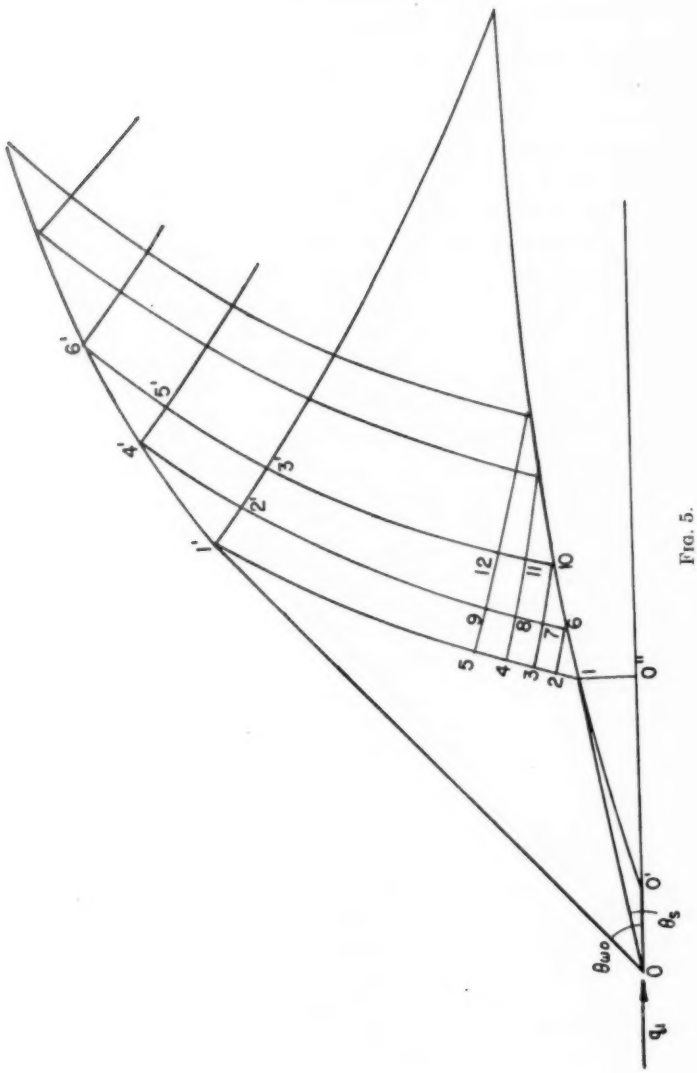

FIG. 5.

Type a: point 2. Then $r = r_1 M(\theta_2)$, $u = 0$, $v = 0$; S may be found by Eq. (32) and the Rankine-Hugoniot equations. This value of the entropy prevails in the whole region bounded by the straight shock $01'$, the contour, and the streamline through $1'$.

Type b: point 10. By Eqs. (16) and (24) $r = f(\theta) = r_2 N(\theta)/N(\theta_7)$, which permits the determination of θ, and then r. Further, by (19) and (25),

$$(v - v_7) + m(u - u_7) + L_m(\theta - \theta_7) = 0,$$

$$ru - f'(\theta)v + rU - f'(\theta)V = 0.$$

Here r and θ are to be taken at 10, and the coefficient L_m is first computed at $u_m = u_7$, $v_m = v_7$. Since $S_\theta = 0$, $L_m = C^{-1}(Du_m + Ev_m)$. The solutions $u^{(1)}$, $v^{(1)}$ of the last two equations may then be improved by computing L_m at $u_m = (u^{(1)} + u_7)/2$, $v_m = (v^{(1)} + v_7)/2$, getting solutions $u^{(2)}$, $v^{(2)}$, and then forming L_m at $u_m = (u^{(2)} + u_7)/2$, $v_m = (v^{(2)} + v_7)/2$, etc., until the desired accuracy has been achieved.

Type c: point $5'$. By Eq. (17) $M(\theta)/N(\theta) = (r_{4'}/r_{3'})M(\theta_{3'})/N(\theta_{4'})$.

Further, $r = r_{3'}M(\theta)/M(\theta_{3'})$. By Eqs. (18) and (19)

$$(v - v_{3'}) + n(u - u_{3'}) + K_m(\theta - \theta_{3'}) = 0,$$

$$(v - v_{4'}) + m(u - u_{4'}) + L_m(\theta - \theta_{4'}) = 0,$$

with K_m and L_m evaluated at $u_m = (u_{4'} + u_{3'})/2$, $v_m = (v_{4'} + v_{3'})/2$, and $S_{\theta,m} = (S_{\theta,4'} + S_{\theta,3'})/2$. Finally, by Eq. (20) $S - S_{3'} = S_{\theta,m}(\theta - \theta_{3'})(1 - mY/X)$. The value of S_θ at $5'$ is obtained by Eq. (23).

Type d: point $6'$. By Eqs. (26) and (13), approximately,

$$\frac{r - r_{4'}}{\theta - \theta_{4'}} = \left(r\frac{q_1 \sin\theta + Y}{q_1 \cos\theta - X}\right)_{4'}, \qquad \frac{r - r_{5'}}{\theta - \theta_{5'}} = (rm)_{5'}.$$

If the solutions of these equations are called $r^{(1)}$ and $\theta^{(1)}$, a better approximation may be obtained by forming the right sides of the equations above at $(r^{(1)} + r_{4'})/2$, $(\theta^{(1)} + \theta_{4'})/2$ and $(r^{(1)} + r_{5'})/2$, $(\theta^{(1)} + \theta_{5'})/2$, respectively.

For the determination of u and v at $6'$ we have by Eqs. (18) and (30)

$$(v - v_{5'}) + n(u - u_{5'}) + K_m(\theta - \theta_{5'}) = 0$$

$$\Gamma u + \Delta v + \Lambda = 0;$$

here we may again use successive approximations, as follows. Using at first in K_m the values of u, v, and S_θ at $5'$ we obtain as solutions of the last two equations $u = u^{(1)}$, $v = v^{(1)}$. Equation (31) will now give $S^{(1)}$, if the value of $dS/d\theta$ is computed by Eq. (33) by means of (27). For the second approximation we use in $K_m u_m = (u^{(1)} + u_{5'})/2$, $v_m = (v^{(1)} + v_{5'})/2$, $S_{\theta,m} = (S_\theta^{(1)} + S_{\theta,5'})/2$, obtain $u^{(2)}$, $v^{(2)}$, $S_\theta^{(2)}$, etc., and continue until the desired accuracy has been achieved.

ASYMPTOTIC EXPANSIONS OF COULOMB WAVE FUNCTIONS*

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Introduction. Coulomb wave functions are used in nuclear physics in problems involving collisions of charged particles. The general picture in nuclear phenomena was clarified considerably by the analysis of the process of alpha particle emission by naturally radioactive elements which was initiated by the work of Gamow and of Condon and Gurney. In the discussions of this problem the large decrease of the Coulomb wave functions which takes place between the nuclear boundary and the high distance edge of the Coulomb barrier is of primary importance. This large decrease is very sensitive to both the height and width of the Coulomb barrier. The main feature of Coulomb wave functions which was important in the early work just mentioned is the critical dependence of the wave function on barrier properties. It was possible to be sure of the qualitative correctness of the explanation and its adequacy in accounting for the Geiger-Nuttall relation without having exact values of Coulomb wave functions. Their approximate representations by the Jeffries-Wentzel-Kramers-Brillouin method showed that alpha particles escape as though they were leaking through the Coulomb barrier in accordance with the Schroedinger equation. In later work it became unsatisfactory to rely on these approximations because the interest in nuclear physics grew and included a new range of experimental facts concerned with light nuclei and new reactions for heavy nuclei and also because the effect of changing the values of the angular momentum assumed for the escape of alpha particles in the uranium, thorium, and actinium families became of importance. For reactions involving light particles it is desirable to have reasonably accurate values of Coulomb wave functions. This is because the decrease of the wave function on passing through the barrier is not an especially dominating effect and has to be considered pretty much on a par with other factors, such as formation of semi-stable states by the bombarded and bombarding particles or the energy dependence of escape probabilities of fragments, which has sometimes no marked relation to Coulomb repulsion. It has also proved possible to draw some reasonably accurate conclusions concerning the magnitude of nuclear forces and the range of distances within which they are effective by the analysis of experiments on the scattering of protons by protons. In this connection, experiments on the scattering of protons by deuterons can be used to give additional information. In the analysis of proton-proton scattering experiments it is necessary to use exact rather than approximate values of Coulomb wave functions. The experiments are being extended in many laboratories to include a wider range of energies, and the tabulations which have been made in the past no longer suffice for the analysis. It is very valuable therefore to have more formulas and expansions of a suitable character for the computation of Coulomb wave functions for new ranges of parameters. Although it would be doubtless possible to obtain all the necessary values by purely computational means, it is also very helpful to have analytic representations which enable a physicist to think about what one might expect without the examination of voluminous tabulations of numbers.

*Received May 27, 1948. The introductory material has been furnished by Professor Gregory Breit of Yale University.

1. **Statement of Problem.** The Computation Laboratory of the National Applied Mathematics Laboratories, National Bureau of Standards is now engaged in the tabulation of the regular solution of the differential equation

$$y'' + \{1 - 2\eta\rho^{-1} - L(L+1)\rho^{-2}\}y = 0 \tag{1.1}$$

for $L = 0$ and for ρ and η in the range from 0 to 10. For physical applications, a solution is desired such that $y = F_L(\rho, \eta) \to \sin[\rho - \eta \ln 2\rho + L\pi/2 + \sigma_L]$ where $\sigma_L = \arg \Gamma(L + 1 + i\eta)$ as $\rho \to \infty$. If we define

$$F_0(\rho, \eta) = C_0\rho\phi_0(\rho, \eta) = C_0\rho \sum_{n=1}^{\infty} a_n(\eta)\rho^{n-1}, \tag{1.2}$$

where C_0 is a normalization factor independent of ρ, it may be shown that the coefficients a_n satisfy the following recurrence relations

$$a_1 = 1, \qquad a_2 = \eta, \qquad n(n-1)a_n = 2\eta a_{n-1} - a_{n-2} \qquad (n = 3, 4, \cdots).$$

The power series (1.2) is satisfactory for computational purposes when ρ and η are small. However, for large values of ρ and η the convergence is slow and a large number of terms are needed. It is the purpose of this article to obtain a number of expansions which may be used for large values of ρ and η.

2. **Asymptotic expansion valid for $\rho \gg \eta$.** When $L = 0$, equation (1.1) becomes

$$y'' + (1 - 2\eta\rho^{-1})y = 0. \tag{2.1}$$

This equation has been studied in considerable detail by T. Sexl[1] and M. Frenkel.[2] The results obtained by Sexl will serve as a starting point for our discussion. Particular solutions of (2.1) may be expressed as contour integrals, namely

$$\begin{aligned} y_1 &= \frac{1}{2\pi i}\int_{C_1} e^{\rho z}(z-i)^{i\eta-1}(z+i)^{-i\eta-1}\,dz, \\ y_2 &= \frac{1}{2\pi i}\int_{C_2} e^{\rho z}(z-i)^{i\eta-1}(z+i)^{-i\eta-1}\,dz, \end{aligned} \tag{2.2}$$

where the paths of integration are shown in the accompanying diagram. It can be shown that

$$\rho\phi_0(\rho, \eta) = e^{\pi\eta}y_1 + y_2\,. \tag{2.3}$$

In order to obtain an expansion in reciprocal powers of ρ Sexl employs the Mellin-Barnes integral representation of a solution of (2.1). We shall obtain this expansion directly from (2.2) and then show how C_0 must be chosen so that the solution $F_0(\rho, \eta)$ should have the desired asymptotic behavior as $\rho \to \infty$.

If we set $(z + i)\rho = v$ in the integral for y_1 we obtain

$$y_1 = \left\{\frac{e^{-i\rho}(2i\rho)^{i\eta}}{(-2i)}\right\}\frac{1}{2\pi i}\int_L e^{v}\left(1 - \frac{v}{2i\rho}\right)^{i\eta-1} v^{-i\eta-1}\,dv, \tag{2.4}$$

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[1] T. Sexl, *Zur Theorie der bei der wellenmechanischen Behandlung des radioaktiven Zerfalls auftretenden Differentialgleichung*, Zeit. für Phys. **56**, (1929).

[2] M. Frenkel, *Die asymptotischer Lösungen der in der Theorie der radioaktiven α-Emission auftretenden Differentialgleichung*, Zeit für Phys. **95**, (May 1935).

the path of integration being a circuit from $-\infty$ around $v = 0$ and then back to $-\infty$. Making use of the relation[3]

$$\left(1 - \frac{v}{2i\rho}\right)^{i\eta-1} = \sum_{r=0}^{s-1} \frac{\Gamma(i\eta)}{r!\Gamma(i\eta - r)} \left(\frac{-v}{2i\rho}\right)^r + R_s , \tag{2.5}$$

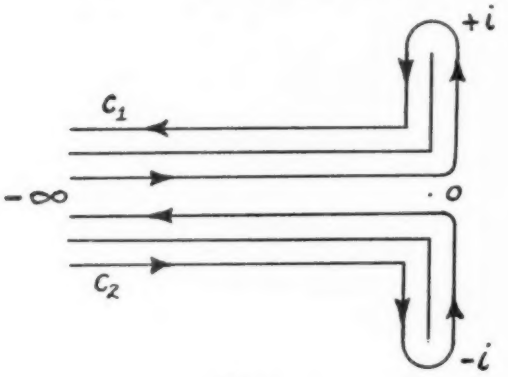

FIG. 1.

where

$$R_s = \frac{\Gamma(i\eta)}{s!\Gamma(i\eta - s)} \left(\frac{-v}{2i\rho}\right)^s \int_0^1 s(1-t)^{s-1}\left(1 - \frac{vt}{2i\rho}\right)^{i\eta-s-1} dt,$$

(2.4) becomes

$$y_1 = \left\{\frac{e^{-i\rho}(2i\rho)^{i\eta}}{(-2i\rho)^s s!\Gamma(i\eta - s)}\right\}\left[\frac{1}{2\pi i}\int_L \sum_{r=0}^{s-1} \frac{(-1)^r\Gamma(i\eta)e^v v^{r-i\eta-1}}{r!\Gamma(i\eta - r)(2i\rho)^r}\, dv + \bar{R}_s\right]$$

where

$$\bar{R}_s = \frac{\Gamma(i\eta)}{(-2i\rho)^s s!\Gamma(i\eta - s)} \frac{1}{2\pi i}\int_L e^v v^{-i\eta-1+s}\, dv \int_0^1 s(1-t)^{s-1}\left(1 - \frac{vt}{2i\rho}\right)^{i\eta-s-1} dt.$$

If in the above relation we invert the order of integration and summation and employ the relation[4]

$$\frac{1}{2\pi i}\int_L e^v v^{-n}\, dv = 1/\Gamma(n),$$

we get finally

$$y_1 = \left\{\frac{e^{-i\rho}(2i\rho)^{i\eta}}{(-2i)}\right\}\left[\sum_{r=0}^{s-1} \frac{(-1)^r\Gamma(i\eta)}{(2i\rho)^r r!\Gamma(i\eta - r)\Gamma(i\eta + 1 - r)} + \bar{R}_s\right] \tag{2.6}$$

The series (2.6) considered as an infinite series is divergent. However, if we truncate the series with the s-th term, it may be shown that $|\bar{R}_s|$ is of the order of $|\rho^{-s}|$ and

[3] T. M. MacRobert, *Functions of a complex variable*, Macmillan and Co., London, 1947, p. 271.

[4] R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, Interscience Publishers, New York, 1943, vol. 1, p. 417.

may be made as small as we please by taking ρ sufficiently large. The proof is entirely similar to that for the Hankel function expansion[5]

The series (2.6) may be rewritten in the form

$$y_1 = \frac{e^{-i\rho}(2i\rho)^{i\eta}}{(-2i)\Gamma(1+i\eta)}\, U(\rho, \eta), \tag{2.7}$$

where

$$U(\rho, \eta) = 1 - \frac{i\eta(i\eta-1)}{2i\rho} + \frac{i\eta(i\eta-1)^2(i\eta-2)}{2!(2i\rho)^2} - \cdots . \tag{2.8}$$

It is clear that to obtain the equivalent result for y_2 we need only change $+i$ to $-i$ in (2.7). We then find

$$y_2 = \frac{e^{i\rho}(-2i\rho)^{-i\eta}}{(2i)\Gamma(1-i\eta)}\, \overline{U}(\rho, \eta) \tag{2.9}$$

where $\overline{U}(\rho, \eta)$ is the complex conjugate of $U(\rho, \eta)$.

Now, let $\Gamma(1+i\eta) = |\Gamma(1+i\eta)|\, e^{i\sigma_0}$ so that (2.7) may be written

$$y_1 = \frac{U(\rho, \eta)e^{-\pi\eta/2}}{(-2i)\,|\Gamma(1+i\eta)|} \exp\{-i(\rho - \eta \ln 2\rho + \sigma_0)\} \tag{2.10}$$

To transform (2.9) in a similar manner we make use of the relation[6]

$$\sigma_0 = \arg\Gamma(1+i\eta) = -\gamma\eta + \sum_{n=1}^{\infty}\left(\frac{\eta}{n} - \arctan\frac{\eta}{n}\right) = -\arg\Gamma(1-i\eta), \tag{2.11}$$

where $\gamma = .577215665\cdots$ is Euler's constant, and the relation[7]

$$|\Gamma(1+i\eta)| = |\Gamma(1-i\eta)| = \left(\frac{2\pi\eta}{e^{\pi\eta} - e^{-\pi\eta}}\right)^{1/2} \tag{2.12}$$

and obtain

$$y_2 = \frac{\overline{U}(\rho, \eta)e^{\pi\eta/2}}{(2i)\,|\Gamma(1-i\eta)|} \exp\{i(\rho - \eta \ln 2\rho + \sigma_0)\}. \tag{2.13}$$

Making use of (2.3), (2.10), and (2.13) we finally get

$$\rho\phi_0(\rho, \eta) = e^{\pi\eta}y_1 + y_2 = \left\{\frac{\overline{U}e^{i\varphi} - Ue^{-i\varphi}}{2i}\right\} \tag{2.14}$$

where $\varphi = \rho - \eta \ln 2\rho + \sigma_0$. Thus, if $U = u + iv$, we get

$$F_0(\rho, \eta) = C_0\rho\phi_0(\rho, \eta) = u\sin\varphi - v\cos\varphi \tag{2.15}$$

where $C_0 = e^{-\pi\eta/2}\,|\Gamma(1+i\eta)|$ and where

$$u = 1 + \frac{\eta}{2\rho} + \frac{5\eta^2 - \eta^4}{2!(2\rho)^2} + \cdots ; \qquad v = \frac{-\eta^2}{2\rho} + \frac{4\eta^3 - 2\eta}{2!(2\rho)^2} - \cdots .$$

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[5] *Ibid*, p. 453.

[6] E. Jahnke and F. Emde, *Tables of functions*, Dover Publications, New York, 1945, p. 10.

[7] T. J. I'A. Bromwich, *An introduction to the theory of infinite series*, Macmillan and Co., London, 1926, p. 474.

It is interesting to note that the expression for $F_0(\rho, \eta)$ can be obtained from (2.1) by elementary methods. Since (2.1) reduces to $y'' + y = 0$ as $\rho \to \infty$, this suggests the substitution

$$y = e^{\mp i\rho}\rho^{\mp k}u(\rho, \eta). \tag{2.16}$$

The differential equation for the function u is

$$u'' + (\pm 2i \mp 2k\rho^{-1})u' + \{(\mp 2ik - 2\eta)\rho^{-1} + k(k \pm 1)\rho^{-2}\}u = 0. \tag{2.17}$$

If we assume the expansion

$$u = 1 + \frac{a_1}{\rho} + \frac{a_2}{\rho^2} + \cdots \tag{2.18}$$

and substitute in (2.17), we obtain

$$(\mp 2ik - 2\eta)\rho^{-1} + \{a_1[\mp 2ik - 2\eta \mp 2i] + k(k \pm 1)\}\rho^{-2}$$

$$+ \{a_2[\mp 2ik - 2\eta \mp 2i] + a_1[k(k \pm 1) \pm 2k]\}\rho^{-3} + \cdots = 0.$$

Thus, if we set $k = \pm i\eta$, the coefficients $a_1, a_2, a_3, \cdots$ can be determined successively. The function $F_0(\rho, \eta)$ is then obtained by taking the proper linear combination of the solutions thus found.

The relation (2.15) provides a convenient method of calculating the zeros of the function $\rho\phi_0(\rho, \eta)$. Setting (2.15) equal to zero, we get

$$\tan \varphi = v(\rho, \eta)/u(\rho, \eta) \tag{2.19}$$

and therefore $\varphi = \rho - \eta \ln 2\rho + \sigma_0 = \arctan v/u + n\pi$($n$, an integer). This relation can be formulated iteratively as a formula for calculating the zeros, namely

$$\rho_s^{(n)} = \eta \ln 2\rho_{s-1}^{(n)} - \sigma_0 + n\pi + \arctan \{v(\rho_{s-1}^{(n)}, \eta)/u(\rho_{s-1}^{(n)}, \eta)\} \tag{2.20}$$

Thus, if $\rho_{s-1}^{(n)}$ represents an approximation to $\rho^{(n)}$, a better approximation can be obtained from (2.20). A short table of these zeros is given below.

η	$n = 1$	$n = 2$	$n = 3$
0.0	3.1416	6.2832	9.4248
0.5	4.4578	7.8928	11.214
1.0	5.8140	9.4746	12.942
1.5	7.1262	10.974	14.567
2.0	8.3957	12.403	16.110
2.5	9.6330	13.786	17.596
3.0	10.846	15.130	19.033

3. Further expansions for large values of ρ or η. Examination of the expansions (2.7) and (2.15) shows that for suitable convergence it is necessary that ρ shall be large with respect to η. In the following development we shall obtain other expansions valid for $\rho < 2\eta$ as well as expansions for $\rho > 2\eta$.

Sexl obtained approximations to the contour integrals y_1 and y_2 given in (2.2) by

the method of saddle points. Taking the appropriate linear combination defined in (2.3) we obtain

$$\rho\phi_0(\rho, \eta) \cong \frac{1}{(8\pi\eta)^{1/2}}\left(\frac{\rho}{2\eta - \rho}\right)^{1/4} \exp\left\{[\rho(2\eta - \rho)]^{1/2} + 2\eta \arcsin (\rho/2\eta)^{1/2}\right\} \quad \text{for} \quad \rho < 2\eta \tag{3.1}$$

$$\rho\phi_0(\rho, \eta) \cong \frac{e^{\pi\eta}}{(2\pi\eta)^{1/2}}\left(\frac{\rho}{2\eta - \rho}\right)^{1/4} \sin [h(\rho, \eta)] \quad \text{for} \quad \rho > 2\eta \tag{3.2}$$

where

$$h(\rho, \eta) = \frac{\pi}{4} + 2\eta \ln\left[\left(\frac{\rho}{2\eta}\right)^{1/2} - \left(\frac{\rho - 2\eta}{2\eta}\right)^{1/2} + [\rho(\rho - 2\eta)]^{1/2}\right]$$

The approximations (3.1) and (3.2) can be obtained directly from the differential equation (2.1) except for the normalizing constants. The method to be described is such that improved approximations can be obtained by elementary methods.

4. Derivation of (3.1). To obtain (3.1) we first put $\rho = 2\eta t$ in (2.1). The resulting equation is

$$y'' + 4\eta^2(1 - t^{-1})y = 0. \tag{4.1}$$

If we now let $y = \exp \varphi$ the differential equation for the function φ is

$$\varphi'^2 + \varphi'' + 4\eta^2(1 - t^{-1}) = 0. \tag{4.2}$$

This equation can be solved if we take for $\varphi(\rho, \eta)$ an expansion of the form

$$\varphi(t, \eta) = (2\eta)g_0(t) + g_1(t) + \frac{1}{(2\eta)}\, g_2(t) + \frac{1}{(2\eta)^2}\, g_3(t) + \cdots . \tag{4.3}$$

Substituting (4.3) into (4.2) and equating coefficients of the successive powers of (2η) to zero, we obtain the following systems of equations

$$\begin{aligned} &g_0'^2 + (1 - t^{-1}) = 0, \\ &g_0'' + 2g_0'g_1' = 0, \\ &g_1'' + g_1'^2 + 2g_0'g_2' = 0, \\ &g_2'' + 2g_1'g_2' + 2g_0'g_3' = 0, \\ &\cdots \end{aligned} \tag{4.4}$$

Solving these systems successively for g_0 , g_1 , g_2 , we find

$$\begin{aligned} g_0 &= \frac{1}{2\eta}\,[\rho(2\eta - \rho)]^{1/2} + \arcsin (\rho/2\eta)^{1/2}, \\ g_1 &= \frac{1}{4} \ln\left(\frac{\rho}{2\eta - \rho}\right), \\ g_2 &= -\left(\frac{\rho}{2\eta - \rho}\right)^{1/2}\left\{\frac{2\rho^2 - 6\eta\rho + 9\eta^2}{12\eta(2\eta - \rho)}\right\}. \end{aligned} \tag{4.5}$$

Further approximations can be obtained by solving (4.4) for g_3 , g_4 , $\cdots$

Comparison of (3.1) with $\exp \varphi$ where φ is defined by (4.3) in conjunction with (4.5) shows that if in (4.3) the terms beyond g_1 are neglected the approximation agrees with (3.1) except for an arbitrary normalizing constant. This suggests using the constant in (3.1) for the approximation just obtained. Thus for $\rho < 2\eta$

$$\rho\phi_0(\rho, \eta) = \frac{1}{(8\pi\eta)^{1/2}} \left(\frac{\rho}{2\eta - \rho}\right)^{1/4} \exp \{\varphi(\rho, \eta)\} \tag{4.6}$$

where

$$\varphi(\rho, \eta) = 2\eta \arcsin \left(\frac{\rho}{2\eta}\right)^{1/2} + [\rho(2\eta - \rho)]^{1/2} - \frac{1}{2\eta} \left(\frac{\rho}{2\eta - \rho}\right)^{1/2} \left\{\frac{2\rho^2 - 6\eta\rho + 9\eta^2}{12\eta(2\eta - \rho)}\right\} + \cdots .$$

5. A second approximation to $\rho\phi_0(\rho, \eta)$ for $\rho < 2\eta$. Let

$$y = N \exp \{\varphi(\rho, \eta)\} \tag{5.1}$$

where N is a constant to be determined subsequently. Substituting (5.1) into (2.1) we obtain

$$\varphi'^2 + \varphi'' + 1 - 2\eta\rho^{-1} = 0. \tag{5.2}$$

Assume for $\varphi(\rho, \eta)$ an expansion of the form

$$\varphi(\rho, \eta) = (2\eta)^{1/2} g_0(\rho) + g_1(\rho) + (2\eta)^{-1/2} g_2(\rho) + \cdots . \tag{5.3}$$

Substituting in (5.2) and equating powers of $(2\eta)^{1/2}$ to zero, we get

$$\begin{aligned} g_0'^2 - \rho^{-1} &= 0, \\ 2g_0'g_1' + g_0'' &= 0, \\ &\cdots \end{aligned} \tag{5.4}$$

These equations may be solved successively, and we obtain

$$\begin{aligned} g_0 &= 2\rho^{1/2}, & g_3 &= \rho^{-1}\left(\frac{\rho^2}{4} - \frac{3}{64}\right), \\ g_1 &= \frac{1}{4} \ln \rho, & g_4 &= -\rho^{-3/2}\left(\frac{\rho^4}{20} + \frac{\rho^2}{32} + \frac{21}{1024}\right). \\ g_2 &= -\rho^{-1/2}\left(\frac{\rho^2}{3} + \frac{3}{16}\right), \end{aligned} \tag{5.5}$$

The expression for $\rho\phi_0(\rho, \eta)$ is thus determined except for the arbitrary normalizing factor. To determine this factor we note that if $\rho \ll 2\eta$, the relation (3.1) becomes

$$\rho\phi_0(\rho, \eta) \cong \frac{1}{(8\pi\eta)^{1/2}} \left(\frac{\rho}{2\eta}\right)^{1/4} \exp [2(2\eta\rho)^{1/2}]. \tag{5.6}$$

Furthermore, if we neglect the terms beyond $g_1(\rho)$ in (5.5) the resulting approximation for $\rho\phi_0(\rho,\eta)$ is the same as (5.6). This suggests choosing $N = (8\pi\eta)^{-1/2}(2\eta)^{-1/4}$ in (5.1). The expression for $\rho\phi_0(\rho, \eta)$ now becomes

$$\rho\phi_0(\rho, \eta) = \frac{1}{(8\pi\eta)^{1/2}}\left(\frac{\rho}{2\eta}\right)^{1/4} \exp\left[\varphi(\rho, \eta)\right] \tag{5.7}$$

where

$$\varphi(\rho, \eta) = 2(2\eta\rho)^{1/2} - \frac{1}{(2\eta\rho)^{1/2}}\left(\frac{\rho^2}{3} + \frac{3}{16}\right) + \frac{1}{2\eta\rho}\left(\frac{\rho^2}{4} - \frac{3}{64}\right)$$

$$- \frac{1}{(2\eta\rho)^{3/2}}\left(\frac{\rho^4}{20} + \frac{\rho^2}{32} + \frac{21}{1024}\right) + \cdots$$

6. A third approximation to $\rho\phi_0(\rho, \eta)$ for $\rho < 2\eta$. The expression (5.7) suggests still another possible form of solution. We assume that

$$\rho\phi_0(\rho, \eta) = \frac{1}{(8\pi\eta)^{1/2}}\left(\frac{\rho}{2\eta}\right)^{1/4} \exp\left[2(2\eta\rho)^{1/2}\right]u(\rho, \eta) \tag{6.1}$$

so that $u(\rho, \eta)$ must be a solution of the differential equation

$$u'' + \left(2(2\eta/\rho)^{1/2} + \frac{1}{2\rho}\right)u' + \left(1 - \frac{3}{16\rho^2}\right)u = 0. \tag{6.2}$$

If we now assume that u may be expressed in the form

$$u = u_0 + \frac{u_1}{(2\eta)^{1/2}} + \frac{u_2}{2\eta} + \frac{u_3}{(2\eta)^{3/2}} + \cdots,$$

we get the following system of equations

$$\frac{2}{\rho^{1/2}}u'_{n+1} + u''_n + \frac{1}{\rho}u'_n + \left(1 - \frac{3}{16\rho^2}\right)u_n = 0, \qquad n = 1, 2, 3, \cdots \tag{6.3}$$

with $u_0 = 1$ determined by the form of (6.1). Solving (6.3) successively we get

$$u_1 = -\rho^{-1/2}\left(\frac{\rho^2}{3} + \frac{3}{16}\right),$$

$$u_2 = \rho^{-1}\left(\frac{\rho^4}{18} + \frac{7\rho^2}{16} - \frac{3}{512}\right), \tag{6.4}$$

$$u_3 = -\rho^{-3/2}\left(\frac{\rho^6}{162} + \frac{89\rho^4}{480} + \frac{179\rho^2}{512} + \frac{13}{8192}\right).$$

7. Approximation to $\rho\phi_0(\rho, \eta)$ for $\rho > 2\eta$. We now proceed to obtain an approximation to $\rho\phi_0(\rho, \eta)$, valid for $\rho > 2\eta$. If in (2.1) we make the substitution $t = (\rho - 2\eta)/2\eta$ the resulting differential equation becomes

$$y'' + \frac{4\eta^2 t}{1 + t}y = 0. \tag{7.1}$$

To derive the solution from this equation, the expression (3.2) suggests that we assume an expansion of the form $y = \exp [H(t, \eta)]$ where

$$H(t, \eta) = \frac{\pi}{4} + 2\eta g_0(t) + g_1(t) + \frac{1}{2\eta} g_2(t) + \cdots . \tag{7.2}$$

The solution of (7.1) thus reduces once more to the solution of a system of equations to be solved successively and we ultimately get

$$\begin{aligned} g_0 &= i \ln \{(t+1)^{1/2} - t^{1/2} + [t(t+1)]^{1/2}\}, \\ g_1 &= \frac{1}{4} \ln \left\{\frac{t+1}{t}\right\}, \\ g_2 &= -i\left(\frac{1+t}{t}\right)^{3/2}\left\{\frac{8t^2+4t+5}{48(1+t)^2}\right\}. \end{aligned} \tag{7.3}$$

The desired approximation to $\rho\phi_0(\rho, \eta)$ corresponds to the imaginary part of $\exp [H(t, \eta)]$. Thus if $t = (\rho - 2\eta)/2\eta$ using the constant in (3.2) we find

$$\rho\phi_0(\rho, \eta) = \frac{e^{\pi\eta}}{(2\pi\eta)^{1/2}} \left(\frac{t+1}{t}\right)^{1/4} \sin [H(t, \eta)] \tag{7.4}$$

where

$$\begin{aligned} H(t, \eta) = \frac{\pi}{4} &+ 2\eta \ln \{(t+1)^{1/2} - t^{1/2} + [t(t+1)]^{1/2}\} \\ &- \frac{1}{2\eta}\left(\frac{1+t}{t}\right)^{3/2}\left\{\frac{8t^2+4t+5}{48(1+t)^2}\right\} + \cdots \end{aligned}$$

8. Development for $2\eta < 1$. We shall now obtain an expansion valid for $2\eta < 1$ and for moderate ρ. Assume that a solution of (2.1) can be expressed in the form

$$y = y_0 + (2\eta)y_1 + (2\eta)^2 y_2 + \cdots . \tag{8.1}$$

Since this function must be identical with $\rho\phi_0(\rho, \eta)$, and since $\rho\phi_0(\rho, \eta) = 0$ and $d/d\rho\{\rho\phi_0(\rho, \eta)\} = 1$ for $\rho = 0$ we shall require that $y(0) = 0$, $y'(0) = 1$, $y_n(0) = 0$, $y_n'(0) = 0$ for $n \geq 1$. Substituting in (2.1) we then get

$$\begin{aligned} y_0'' + y_0 &= 0, \\ y_n'' + y_n &= \frac{1}{\rho} y_{n-1} \qquad \text{for} \qquad n = 1, 2, 3, \cdots . \end{aligned} \tag{8.2}$$

Solving this system of equations, we get

$$\begin{aligned} y_0 &= \sin \rho = \rho - \frac{\rho^3}{3!} + \frac{\rho^5}{5!} - \cdots , \\ y_1 &= a_2 \frac{\rho^2}{2!} - a_4 \frac{\rho^4}{4!} + \cdots , \\ a_2 &= 1, \qquad a_4 = a_2 + 1/3, \qquad a_6 = a_4 + 1/5, \cdots \end{aligned} \tag{8.3}$$

$$y_2 = b_3 \frac{\rho^3}{3!} - b_5 \frac{\rho^5}{5!} + \cdots ,$$

$$b_3 = 1/2\, a_2 , \qquad b_5 = b_3 + 1/4\, a_4 , \qquad b_7 = b_5 + 1/6\, a_6 , \cdots$$

$$y_3 = c_4 \frac{\rho^4}{4!} - c_6 \frac{\rho^6}{6!} + \cdots ,$$

$$c_4 = 1/3\, b_3 , \qquad c_6 = c_4 + 1/5\, b_5 , \qquad c_8 = c_6 + 1/7\, b_7 , \cdots .$$

The law of formation of the successive y_n is straightforward.

9. Approximation in terms of Bessel functions for large η. A method similar to that employed in (8.1) may be used to obtain an expansion valid for large values of η and for moderate ρ. We first make the substitution $t = 2\eta\rho$ in (2.1) and obtain

$$y'' - \frac{1}{t} y = -\frac{1}{4\eta^2} y. \tag{9.1}$$

Assume an expansion of the form

$$y = k\left\{y_0(t) - \frac{1}{4\eta^2} y_1(t) + \frac{1}{(4\eta^2)^2} y_2(t) - \cdots\right\}, \tag{9.2}$$

where k is a constant to be chosen so that $y'(0) = 1$. Substituting (9.2) in (9.1) we then obtain

$$y_0'' - t^{-1} y_0 = 0, \tag{9.3}$$

$$y_n'' - t^{-1} y_n = -y_{n-1} , \qquad (n = 1, 2, \cdots). \tag{9.4}$$

The solution of (9.3) which is regular at $t = 0$ is

$$y_0 = t^{1/2} I_1(2t^{1/2}) = \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!(n+1)!}. \tag{9.5}$$

The equations (9.4) may be solved successively, and we find for y_1 and y_2

$$y_1 = \sum_{n=3}^{\infty} \frac{n^3 - 3n^2 + 2n + 12}{3} \frac{t^n}{(n-1)!n!} = \frac{2t^3}{2!3!} + \frac{8t^4}{3!4!} + \frac{20t^5}{4!5!} + \cdots ,$$

$$y_2 = \frac{24t^5}{4!5!} + \frac{184t^6}{5!6!} + \frac{784t^7}{6!7!} + \frac{2864t^8}{7!8!} + \frac{6784t^9}{8!9!} + \cdots . \tag{9.6}$$

It is clear from (9.5) that in order that the expansion (9.2) shall have the same slope as $\rho\phi_0(\rho, \eta)$ at $\rho = 0$ that we must have $k = 1/2\eta$.

It should be mentioned that several of the above methods may be employed to derive expansions of (1.1) for integral values of L.

COMPLEMENTARY MINIMUM PRINCIPLES FOR AN ELASTIC-PLASTIC MATERIAL[1]

BY

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1. Introduction. This paper is concerned with the mechanical behavior of bodies consisting of an isotropic elastic-plastic material which obeys the stress-strain law of Prandtl and Reuss.[2] This inviscid material has a sharply defined yield point and does not exhibit work hardening. To simplify mathematical work, we shall moreover assume it to be incompressible in the elastic as well as the plastic range.

For the elastic range, Hooke's law is adopted in the form which applies to an incompressible elastic solid. In the plastic range, it is assumed that (i) the stress components satisfy the flow conditions, and (ii) the components of the plastic strain rates have to each other the same ratios as the components of the stress deviation. The stress-strain law derived from these assumptions furnishes a unique rate of change of the stress deviation when the stress and the strain rates are given.

Since the stress-strain law of Prandtl and Reuss is differential in form and is not equivalent to any finite relation between stress and strain, it is of the flow type and not the deformation type.[3] As Prager[4] has pointed out, the typical boundary value problems of a plastic flow theory suppose that the state of stress is known throughout the body under consideration and prescribe either the velocities on the surface or the rates of change of the surface stresses, asking in both cases for the rates of change of the stress components in the interior. This paper is concerned with the derivation of two minimum principles relating to the mixed boundary value problem where the velocities are given on part of the surface and the rates of change of the surface stresses on the rest. For the Prandtl-Reuss theory these principles play a role analogous to that played in the theory of elasticity by the principle of Castigliano and the principle of minimum potential energy.

2. Preliminary definitions and relations. We let σ_{ij} and ϵ_{ij} $(i, j = 1, 2, 3)$ denote the components of the tensors of stress and strain and define the *stress deviation* as

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} , \tag{2.1}$$

where the usual summation convention of the tensor calculus is used, and δ_{ij} is the Kronecker delta. On account of the assumed incompressibility of the material

$$\epsilon_{kk} = 0. \tag{2.2}$$

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[1]Received May 27, 1948. The results presented in this paper were obtained in the course of research conducted under a contract sponsored jointly by the Office of Naval Research and the Bureau of Ships.

[2]L. Prandtl, *Spannungsverteilung in plastischen Körpern*, Proc. 1st Internat. Congr. Appl. Mech., Delft, 1924, pp. 43-54; E. Reuss, *Berücksichtigung der elastischen Formänderungen in der Plastizitätstheorie*, Z. angew. Math. Mech. **10**, 266-274 (1930).

[3]This terminology was introduced by A. A. Ilyushin in his paper *Relation between the theory of Saint Venant-Lévy-Mises and the theory of small elastic plastic deformations*, (Russian with English summary), Prikl. Mat. Mekh. **9**, 207-218 (1946).

[4]W. Prager, *Fundamental theorems of a new mathematical theory of plasticity*, Duke Math. Journal **9**, 228-233 (1942).

In working with the equations of any flow theory, it is convenient to replace differentials by the corresponding derivatives (denoted by primes) with respect to a parameter t. For convenience, we speak of these derivatives as rates, although the parameter t with respect to which these rates are taken need not be time. Since all equations which we write are homogeneous in these derivatives, there is no viscosity effect, and replacing the "time" t by any function which increases monotonically with t, e. g. by an angle of twist, a torque, or a load, which increases monotonically with t, leaves the equations unchanged.

In the Prandtl-Reuss theory the condition for the occurrence of *plastic flow* is

$$J_2 = k^2, \tag{2.3}$$

where

$$J_2 = \tfrac{1}{2}\, s_{ij}s_{ij} \tag{2.4}$$

denotes the second invariant of the stress deviation, and k denotes the yield stress in pure shear. The sign of the quantity

$$J_2' = s_{ij}s_{ij}' \tag{2.5}$$

is used as the criterion for *loading*. Thus, $J_2' > 0$ indicates loading, and $J_2' < 0$ unloading. The *stress-strain relations* for this theory are

$$\epsilon_{ij}' = \frac{1}{2G_0}\, s_{ij}' + \mu s_{ij}\ , \tag{2.6}$$

where G_0 is the elastic shear modulus, and

$$\mu \equiv \begin{cases} 0, & \text{wherever } J_2 < k^2, \quad \text{or } J_2 = k^2 \quad \text{but } J_2' < 0, \\ \dfrac{s_{pq}\epsilon_{pq}'}{2k^2}, & \text{wherever } J_2 = k^2 \quad \text{and} \quad J_2' = 0. \end{cases} \tag{2.7}$$

The strain rates are given in terms of the velocities u_i' by the equations

$$\epsilon_{ij}' = \tfrac{1}{2}(u_{i,j}' + u_{j,i}')$$

where we have assumed that the displacements u_i and their derivatives with respect to space and t are small. Physically we note that the quantity $2k^2\mu$ represents the rate at which plastic work is being done. Since plastic work cannot be reclaimed, i.e. since the process of plastic deformation is irreversible, we have

$$\mu \geq 0. \tag{2.8}$$

This interpretation of μ follows from (2.6), (2.7), the assumption of incompressibility and the fact that (2.6) represents a decomposition of ϵ_{ij}' into elastic strain rate and plastic strain rate.

In the following we shall consider a body consisting of an incompressible Prandtl-Reuss material which occupies a volume V in space bounded by a surface S. We shall not be concerned with the specific character of V and S and hence shall assume only that they are sufficiently regular so that (1) solutions exist to the problems posed and (2) the divergence theorem for the transformation of volume into surface integrals remains valid.

We shall assume for a fixed "instant" t that (1) the state of stress σ_{ij} is known throughout the body, (2) the rate of change T_i' of the surface stress vector T_i is prescribed at each point of a portion S_1 of the surface S, (3) the velocity u_i' is prescribed at each point of the remaining portion S_2 of S. The minimum principles we shall formulate will be for the instantaneous stress rates and strain rates throughout the interior of the body.

3. Minimum principle for the stress rates. We consider first the restrictions imposed on the stress rates σ_{ij}' by the *conditions of equilibrium*. Assuming for simplicity that there are no body forces acting, the equation[5]

$$\sigma_{ij,j}' = 0 \tag{3.1}$$

must be everywhere satisfied except at most at the points of a surface Σ separating an elastic region, $J_2 < k^2$, from a plastic region, $J_2 = k^2$. Across such a surface we may encounter discontinuities in individual components of σ_{ij}', but the stress rate vector acting across each element $d\Sigma$ of the surface must be continuous for equilibrium. Thus, if n_j denotes the normal to $d\Sigma$, and if we denote quantities evaluated on one side of $d\Sigma$ by the superscript (1), and quantities on the other side by the superscript (2), it is not necessary that

$$\sigma_{ij}'^{(1)} = \sigma_{ij}'^{(2)}, \qquad i, j = 1, 2, 3$$

but

$$\sigma_{ij}'^{(1)} n_j = \sigma_{ij}'^{(2)} n_j .$$

must be satisfied.

As an example we note the case of torsion of a circular cylinder. Let τ denote the circumferential shearing stress and consider the boundary $r = \rho$ between the elastic and plastic portions. Then, assuming $\theta' > 0$, where θ is the angle of twist, we have $\tau' = 0$ for $r \geq \rho$, since $\tau = k$ in the plastic region, whereas in the elastic portion, $r < \rho$. Now, τ' is a linear function of r, increasing from 0 at $r = 0$ to a maximum as r approaches ρ. Thus, τ' is discontinuous as we cross the surface $r = \rho$. However, the stress transmitted across an element of $r = \rho$ is zero computed from either side, and hence continuous.

In the following, a set of stress rates σ_{ij}' defined throughout the body will be called a *solution* if it satisfies (1) the conditions of equilibrium, (2) the boundary conditions on S_1, (3) the condition that

$$J_2' = s_{ij} s_{ij}' \leq 0 \tag{3.2}$$

throughout the plastic region $J_2 = k^2$, where s_{ij}' denotes the deviation of the σ_{ij}', and (4) the condition that there exists a set of strain rates ϵ_{ij}' (not necessarily unique) satisfying compatibility, incompressibility and the boundary conditions on S_2, which together with the s_{ij}' and the s_{ij} satisfy the stress-strain law of Prandtl and Reuss, Eq. (2.6).

It should be noted that σ_{ij} and hence s_{ij} is known only as a function of position at the instant t and so cannot, of course, simply be differentiated with respect to t to yield σ_{ij}' and s_{ij}'. However, knowledge of s_{ij} as a function of position is sufficient to determine

[5] A subscript following a comma denotes differentiation with respect to the corresponding variable, e.g. $\sigma'_{ij,j} = \partial \sigma'_{ij} / \partial x_j$.

the region $J_2 = k^2$ and to verify in this region for any test solution σ'_{ij} whether or not $s_{ij}s'_{ij} \leq 0$, i.e. whether (3.2) is satisfied.

Let σ'^*_{ij} be an artificial system of stress rates, defined throughout the body, which satisfy (1) the conditions of equilibrium, (2) the boundary conditions on S_1 and (3) the condition that

$$s_{ij}s'^*_{ij} \leq 0 \tag{3.3}$$

throughout the plastic region $J_2 = k^2$, where s'^*_{ij} denotes the deviation of the σ'^*_{ij} . The condition (3.3) eliminates from consideration stress rates which would act to increase J_2 in the plastic region. A system of stress rates σ'^*_{ij} satisfying the above conditions will be called an *admissible* state of stress rates or simply an admissible state.

MINIMUM PRINCIPLE. *If σ'_{ij} is a solution and σ'^*_{ij} is an arbitrary admissible state, then*

$$I(\sigma'_{ij}) \leq I(\sigma'^*_{ij}), \tag{3.4}$$

where I is defined by

$$I(\sigma'^*_{ij}) = \frac{1}{4G_0}\int_V s'^*_{ij}s'^*_{ij}\,dv - \int_{S_2} T'^*_i u'_i\,dS_2\,. \tag{3.5}$$

Equality holds in (3.4) *if and only if* $s'^*_{ij} = s'_{ij}$.

From this principle which states that I is rendered an absolute minimum by a solution, it follows that if a solution σ'_{ij} exists, it is unique in the sense that its deviatoric part s'_{ij} is unique. This solution may be spoken of as the *natural state* of stress rates.

We note that *this principle is not restricted in any way to loading.* Indeed, under the general boundary conditions considered, unloading may conceivably occur anywhere in the body including the already plastic as well as the still elastic portions.

Proof. We first note that

$$\int_V \sigma'^*_{ij}\epsilon'_{ij}\,dv = \int_S T'^*_i u'_i\,dS. \tag{3.6}$$

where $T'^*_i \equiv \sigma'^*_{ij}n_j$ and n_j are the components of the unit normal to S. This is proved in the usual manner using the equations $\sigma'^*_{ij,j} = 0$ and $\epsilon'_{ij} = 1/2\,(u'_{i,j} + u'_{j,i})$ and the divergence theorem to accomplish the transformation from a volume to a surface integral. Care must be taken, however, since the σ'^*_{ij} are not necessarily continuous across a surface within the body which separates a subregion $J_2 < k^2$ from a subregion $J_2 = k^2$. However, the result may be proved for each subregion (assuming each is sufficiently regular) and the results added. On account of the continuity of the stress rate vector across surfaces common to two such subregions the contributions over these surfaces add to zero and we are left with the integral over S.

In view of the assumed incompressibility we may replace (3.6) by

$$\int_V s'^*_{ij}\epsilon'_{ij}\,dv = \int_S T'^*_i u'_i\,dS. \tag{3.7}$$

Similarly, for the solution σ'_{ij}

$$\int_V s'_{ij}\epsilon'_{ij}\,dv = \int_S T'_i u'_i\,dS. \tag{3.8}$$

Subtracting, and substituting for ϵ'_{ij} from (2.6), we obtain

$$\int_V (s'^*_{ij} - s'_{ij})\left(\frac{1}{2G_0}\, s'_{ij} + \mu s_{ij}\right) dv = \int_{S_2} (T'^*_i - T'_i)u'_i \, dS_2 ,$$

since over S_1 we have $T'^*_i = T'_i$. Thus,

$$\frac{1}{2G_0}\int_V (s'^*_{ij} - s'_{ij})s'_{ij}\, dv = -\int_V \mu s_{ij}(s'^*_{ij} - s'_{ij})\, dv + \int_{S_2} (T'^*_i - T'_i)u'_i \, dS_2 . \tag{3.9}$$

From the definition of μ in (2.7), it follows that

$$\mu s_{ij}s'_{ij} = \mu J'_2 = 0 \tag{3.10}$$

throughout the body. Accordingly, (3.9) becomes

$$\frac{1}{2G_0}\int_V s'_{ij}\Delta s'_{ij}\, dv = -\int_V \mu s_{ij}s'^*_{ij}\, dv + \int_{S_2} u'_i \Delta T'_i \, dS_2 , \tag{3.11}$$

where

$$\Delta s'_{ij} = s'^*_{ij} - s'_{ij} , \qquad \Delta T'_i = T'^*_i - T'_i . \tag{3.12}$$

Substituting from (3.12) for s'^*_{ij} and T'^*_i into (3.5) we find that

$$I(\sigma'^*_{ij}) = \frac{1}{4G_0}\int_V s'_{ij}s'_{ij}\, dv - \int_{S_2} T'_i u'_i \, dS_2 + \frac{1}{2G_0}\int_V s'_{ij}\Delta s'_{ij}\, dv - \int_{S_2} u'_i \Delta T'_i \, dS_2 + \frac{1}{4G_0}\int_V \Delta s'_{ij}\Delta s'_{ij}\, dv. \tag{3.13}$$

The first two terms on the right-hand side of (3.13) constitute $I(\sigma'_{ij})$ and the third term can be eliminated by means of (3.11); thus,

$$I(\sigma'^*_{ij}) - I(\sigma'_{ij}) = -\int_V \mu s_{ij}s'^*_{ij}\, dv + \frac{1}{4G_0}\int_V \Delta s'_{ij}\Delta s'_{ij}\, dv. \tag{3.14}$$

The last term in (3.14) is always positive, vanishing if, and only if, $s'^*_{ij} = s'_{ij}$. Since $\mu \geq 0$ and, by (3.13), $s_{ij}s'^*_{ij} \leq 0$ wherever $\mu > 0$, it follows that the first term on the right-hand side of (3.14) is also positive or zero; this term vanishes for $s'^*_{ij} = s'_{ij}$ since $s_{ij}s'_{ij} = J'_2 = 0$ for $\mu > 0$. The minimum principle is thereby proved.

We remark that in the case where the stress rate vector is everywhere prescribed on the boundary, this principle reduces to the statement that the quantity

$$\frac{1}{4G_0}\int_V s'^*_{ij}s'^*_{ij}\, dv \tag{3.15}$$

is an absolute minimum for the natural state. The integral (3.15), in effect, measures the total intensity of s'^*_{ij} , the non-hydrostatic and hence "plastically active" part of the stress rates σ'^*_{ij} . Thus, the principle given may be termed one of *minimum stress rate intensity*.

It is interesting to compare this principle of minimum stress rate intensity with the Castigliano principle of *elasticity* which is one of minimum stress intensity or stress "energy". Roughly speaking, the former principle deals with minimizing the second invariant

$$L_2 = \tfrac{1}{2} s'_{ij}s'_{ij}$$

of the stress rate deviation, while the latter principle for an incompressible elastic solid deals with minimizing the second invariant

$$J_2 = \tfrac{1}{2} s_{ij}s_{ij}$$

of the stress deviation. The Castigliano principle, however, can be easily transformed into an "incremental" principle for the increments of loads and the increments of stress they produce. This incremental principle is the counterpart of the principle discussed here for plastic bodies. We note that in the elastic case the increments of stress are independent of the already existing stress, while in the plastic case they are not.

4. Minimum principle for the strain rates. We now restrict our attention to the case where the effect of the prescribed stress rates T'_i on S_1 and the prescribed velocities u'_i on S_2 turn out to be such that unloading does not occur in the plastic region, i.e. $J'_2 = 0$ wherever $J_2 = k^2$. With this restriction we prove a minimum principle for the strain rates for the Prandtl-Reuss material.

A set of strain rates ϵ'_{ij} will be called a *solution* provided that it satisfies compatibility, incompressibility, the boundary conditions on S_2 and determines, together with the σ_{ij} (which are assumed to be known as in Sec. 3), through (2.6), an equilibrium set of stress rates which satisfy the boundary conditions on S_1 .[6] We assume that such a solution ϵ'_{ij} , not necessarily unique, exists. Let ϵ'^*_{ij} denote any artificial system of strain rates which satisfy compatibility, incompressibility and whose corresponding velocities u'^*_i satisfy the boundary conditions on S_2 ; such a system of strain rates will be called an admissible state.

MINIMUM PRINCIPLE. *If ϵ'_{ij} is a solution and ϵ'^*_{ij} is an arbitrary admissible state, then*

$$J(\epsilon'_{ij}) \leq J(\epsilon'^*_{ij}) \tag{4.1}$$

where J is defined by

$$J(\epsilon'^*_{ij}) = G_0 \int_V [\epsilon'^*_{ij}\epsilon'^*_{ij} - \mu^* s_{ij}\epsilon'^*_{ij}]\, dv - \int_{S_1} T'_i u'^*_i\, dS_1 , \tag{4.2}$$

[6] Actually only the stress rate deviations s'_{ij} are determined by (2.6). To obtain the condition which these must satisfy for equilibrium we rewrite (3.1) in the form

$$s'_{ij,j} - p'_{,i} = 0$$

where $p' = -\sigma'_{kk}/3$ is the mean pressure rate. Eliminating p' we find that

$$s'_{ij,jk} - s'_{kj,ji} = 0.$$

Provided that the s'_{ij} satisfy this equation, we may compute the gradient of the mean pressure rate from the preceding equation. The mean pressure rate, and hence $\sigma'_{ij} = s'_{ij} - p'\delta_{ij}$, is thereby determined to within a constant rate of hydrostatic pressure. It may then be checked whether the boundary conditions on S_1 are satisfied (to within a constant rate of hydrostatic pressure).

and

$$\mu^* \equiv \begin{cases} 0, & \text{wherever } J_2 < k^2, \\ \dfrac{s_{pq}\epsilon'^*_{pq}}{2k^2}, & \text{wherever } J_2 = k^2. \end{cases} \tag{4.3}$$

Equality holds in (4.2) *if and only if* ϵ'^*_{ij} *is a solution.*

Proof. From (2.6) and (2.7) we have

$$s'_{ij} = 2G_0(\epsilon'_{ij} - \mu s_{ij}), \tag{4.4}$$

where

$$\mu \equiv \begin{cases} 0, & \text{wherever } J_2 < k^2, \\ \dfrac{s_{pq}\epsilon'_{pq}}{2k^2}, & \text{wherever } J_2 = k^2, \end{cases} \tag{4.5}$$

it having been assumed that there is no region for which $J_2 = k^2$ while $J'_2 < 0$. We note then, by (4.3) and (4.5), that

$$\mu^* = \mu \equiv 0, \qquad \text{wherever} \qquad J_2 < k^2. \tag{4.6}$$

From (4.4) it follows that

$$\int_V s'_{ij}(\epsilon'^*_{ij} - \epsilon'_{ij})\, dv = 2G_0 \int_V (\epsilon'_{ij} - \mu s_{ij})(\epsilon'^*_{ij} - \epsilon'_{ij})\, dv. \tag{4.7}$$

At the same time,

$$\int_V s'_{ij}(\epsilon'^*_{ij} - \epsilon'_{ij})\, dv = \int_V \sigma'_{ij}(\epsilon'^*_{ij} - \epsilon'_{ij})\, dv = \int_{S_1} T'_i(u'^*_i - u'_i)\, dS_1 , \tag{4.8}$$

since both ϵ'^*_{ij} and ϵ'_{ij} satisfy compatibility and correspond to velocities u'^*_i and u'_i which are identical over part S_2 of the boundary. Equating the expressions in (4.7) and (4.8) and introducing the notations

$$\begin{aligned} \Delta\epsilon'_{ij} &= \epsilon'^*_{ij} - \epsilon'_{ij} \\ \Delta u'_i &= u'^*_i - u'_i \end{aligned} \tag{4.9}$$

we find that

$$2G_0 \int_V \epsilon'_{ij}\Delta\epsilon'_{ij}\, dv = 2G_0 \int_V \mu s_{ij}\Delta\epsilon'_{ij}\, dv + \int_{S_1} T'_i \Delta u'_i\, dS_1 . \tag{4.10}$$

Returning to (4.2), we must now substitute for ϵ'^*_{ij} and u'^*_i from (4.9) and expand the expression obtained. We note first, that by virtue of (4.3), (4.5), and the first equation in (4.9) we may write

$$\mu' s_{ij}\epsilon'^*_{ij} = \mu s_{ij}\epsilon'_{ij} + \frac{\tau}{2k^2}, \tag{4.11}$$

where we define

$$\tau \equiv \begin{cases} 0, & \text{wherever } J_2 < k^2, \\ 2(s_{pq}\epsilon'_{pq})(s_{ij}\Delta\epsilon'_{ij}) + (s_{pq}\Delta\epsilon'_{pq})(s_{ij}\Delta\epsilon'_{ij}), & \text{wherever } J_2 = k^2. \end{cases} \tag{4.12}$$

Carrying out the expansion of (4.2) now, using (4.9) and (4.12) we obtain

$$J(\epsilon_{ij}'^{*}) = G_0 \int_V \left[\epsilon_{ij}'\epsilon_{ij}' + 2\epsilon_{ij}'\Delta\epsilon_{ij}' + \Delta\epsilon_{ij}'\Delta\epsilon_{ij}' - \mu s_{ij}\epsilon_{ij}' - \frac{\tau}{2k^2} \right] dv - \int_{S_1} T_i' u_i' \, dS_1 - \int_{S_1} T_i' \Delta u_i' \, dS_1 ,$$

or, introducing $J(\epsilon_{ij}')$ on the right-hand side of this equation and substituting from (4.10)

$$J(\epsilon_{ij}'^{*}) - J(\epsilon_{ij}') = G_0 \int_V \left(2\mu s_{ij}\Delta\epsilon_{ij}' + \Delta\epsilon_{ij}'\Delta\epsilon_{ij}' - \frac{\tau}{2k^2} \right) dv. \tag{4.13}$$

We now examine the integrand in (4.13). Over the region $J_2 < k^2$ it reduces to $\Delta\epsilon_{ij}'\Delta\epsilon_{ij}'$. Hence, over this region the integral in (4.13) is positive, becoming zero if, and only if, $\epsilon_{ij}'^{*} = \epsilon_{ij}'$. Over the region $J_2 = k^2$, we have, upon substituting for μ and τ from (4.5) and (4.12), the following expression for the integrand:

$$\Delta\epsilon_{ij}'\Delta\epsilon_{ij}' - \frac{1}{2k^2}(s_{pq}\Delta\epsilon_{pq}')(s_{ij}\Delta\epsilon_{ij}') = \frac{1}{2k^2}(s_{pq}s_{pq}\Delta\epsilon_{ij}'\Delta\epsilon_{ij}' - s_{pq}\Delta\epsilon_{pq}'s_{ij}\Delta\epsilon_{ij}') \geq 0, \tag{4.14}$$

which is non-negative by Schwartz' inequality for sums. This proves (4.1) and hence that ϵ_{ij}' renders J a minimum.

It remains to consider the question of uniqueness. If in (4.1) we have equality so that

$$J(\epsilon_{ij}'^{*}) = J(\epsilon_{ij}'),$$

then, by what has already been said, it is necessary that

$$\epsilon_{ij}'^{*} \equiv \epsilon_{ij}' , \qquad \text{in the region } J_2 < k^2,$$

i.e. we have uniqueness in the elastic region. Moreover, (4.14) must reduce to an equality, which is the case if, and only if, between the qualities s_{ij} and $\Delta\epsilon_{ij}' = \epsilon_{ij}'^{*} - \epsilon_{ij}'$ there holds the relation[7]

$$\epsilon_{ij}'^{*} - \epsilon_{ij}' = f s_{ij} , \qquad \text{wherever } J_2 = k^2, \tag{4.15}$$

where f is a scalar function of position. To prove that the solution ϵ_{ij}' is unique would be to prove that $f \equiv 0$ throughout the region $J_2 = k^2$. This, the minimum principle does not tell us. Indeed, given s_{ij} and s_{ij}' , the stress-strain relations (4.4) do not by themselves uniquely determine the strain-rates ϵ_{ij}' . Thus, it is by no means clear that the problem has a unique solution. However, the minimum principle does provide us with the statement (4.15). From this we shall at once deduce that if $f \not\equiv 0$, then $\epsilon_{ij}'^{*}$ must be a solution which is distinct from ϵ_{ij}' . Hence, in any event, if for a given set of $\epsilon_{ij}'^{*}$, J attains its minimum value, then $\epsilon_{ij}'^{*}$ is a solution. To do this, we need only show that $\epsilon_{ij}'^{*}$ satisfies (4.4), where s_{ij} and s_{ij}' remain unchanged. Adding $f s_{ij}$ to both sides of (2.6), which is the same equation as (4.4), we have, by (4.15)

$$\epsilon_{ij}'^{*} = \frac{1}{2G_0} s_{ij}' + (f + \mu) s_{ij} , \tag{4.16}$$

[7]The author is indebted to W. Prager for pointing out Eq. (4.15) and calling his attention to the question of the uniqueness of the strain rates.

and we need only consider the region $J_2 = k^2$. From (4.15) it follows that

$$fs_{ij}s_{ij} = f2k^2 = s_{ij}(\epsilon'^*_{ij} - \epsilon'_{ij})$$

so that by (4.5)

$$f = \frac{s_{ij}\epsilon'^*_{ij}}{2k^2} - \mu. \tag{4.17}$$

Substituting (4.17) into (4.16) completes the argument. The converse statement that, if ϵ'^*_{ij} is a solution of the problem which is distinct from ϵ'_{ij}, then $J(\epsilon'^*_{ij}) = J(\epsilon'_{ij})$, follows by a reversal of the above reasoning.

In the next section we shall consider the relation between the two minimum principles which have been given. However, we may notice here that the principle just proved, may be considered as a *principle of minimum complementary (stress-rate) intensity.* To illustrate the meaning of the term complementary we note that from (2.6) we have

$$\epsilon'_{ij}\epsilon'_{ij} = \frac{1}{4G_0^2}\, s'_{ij}s'_{ij} + \frac{\mu}{G_0}\, s'_{ij}s_{ij} + \mu^2 s_{ij}s_{ij}\,,$$

and, therefore, by (2.7) and (3.10),

$$\frac{1}{4G_0}\, s'_{ij}s'_{ij} = G_0(\epsilon'_{ij}\epsilon'_{ij} - \mu s_{ij}\epsilon'_{ij}). \tag{4.18}$$

On the left-hand side of (4.18) we have the stress rate intensity, on the right-hand side, the complementary quantity in the strain rates which is a minimum (under appropriate boundary conditions) by the principle of this section.

5. Relation between the principles. We find the minimum value of I of (3.5) by evaluating $I(\sigma'_{ij})$ where σ'_{ij} is the natural state, i.e. the solution for the stress rates. Substituting from (4.4) into (3.5) yields

$$I(\sigma'_{ij}) = \frac{1}{2}\int_V s'_{ij}(\epsilon'_{ij} - \mu s_{ij})\, dv - \int_{S_2} T'_i u'_i\, dS_2\,. \tag{5.1}$$

Taking into account (3.8) and (3.10) we obtain

$$I(\sigma'_{ij}) = \frac{1}{2}\left[\int_{S_1} T'_i u'_i\, dS_1 - \int_{S_2} T'_i u'_i\, dS_2\right] \tag{5.2}$$

as the minimum value of I.

On the other hand, substituting from (2.6) into $J(\epsilon'_{ij})$ as defined by (4.2), where ϵ'_{ij} is any solution for the strain rates, gives

$$J(\epsilon'_{ij}) = \frac{1}{2}\int_V \epsilon'_{ij}s'_{ij}\, dv - \int_{S_1} T'_i u'_i\, dS_1 \tag{5.3}$$

which, by (3.8), yields

$$J(\epsilon'_{ij}) = \frac{1}{2}\left[\int_{S_2} T'_i u'_i\, dS_2 - \int_{S_1} T'_i u'_i\, dS_1\right] \tag{5.4}$$

for the minimum value of the integral J. Comparing (5.2) and (5.4) we have the following result.[8]

[8]Here we must restrict ourselves to the case where $J'_2 = 0$ for $J_2 = k^2$, which is the one covered by the principle for the strain rates.

For the natural state of stress rates σ'_{ij} and any solution ϵ'_{ij}

$$I(\sigma'_{ij}) = -J(\epsilon'_{ij}) = \frac{1}{2}\left[\int_{S_1} T'_i u'_i \, dS_1 - \int_{S_2} T'_i u'_i \, dS_2\right] \tag{5.5}$$

In particular, for the case of stress rates everywhere prescribed on the boundary, i.e. S_2 vacuous,

$$\frac{1}{4G_0}\int_V s'_{ij}s'_{ij} \, dv = G_0 \int_V (\epsilon'_{ij}\epsilon'_{ij} - \mu s_{ij}\epsilon'_{ij}) \, dv = \frac{1}{2}\int_S T'_i u'_i \, dS. \tag{5.6}$$

Both (5.5) and (5.6) could, of course, be deduced directly from (4.18). Relation (5.6), in the case of elasticity, is equivalent to the familiar theorem that for the natural state, the stress energy equals the strain energy equals the work done by the external forces. The present statement, which is the one valid for elastic-plastic behavior, relates the stress rate intensity to the "work" done over the surface S by the stress rate vector T'_i on the velocity vector u'_i. To gain this interpretation it is more satisfactory to speak in terms of the differentials of stress and displacement, for then the integral becomes

$$\frac{1}{2}\int_S dT_i \, du_i \, dS$$

which properly represents the work done by the dT_i on the du_i.

Relations (3.4), (4.1) and (5.5) can be combined to yield inequalities which furnish upper and lower bounds for the integral in (5.5). A direct method based on these inequalities of constructing approximate solutions to certain types of elastic-plastic problems will be considered in a later paper. Such methods have already been discussed in elasticity by Prager and Synge.[9]

6. Final remarks. In the history of the theories of plasticity various extremum principles have been formulated, of which it is relevant to mention a few. We may subdivide the various theories into theories of deformation and theories of flow (see footnote 3). The present principles appear to be the first which have been given for a theory of *flow* pertaining to an elastic-plastic material. Minimum principles for stress rates and strain rates for a flow theory pertaining to strain hardening materials exhibiting a continuous transition from the elastic to the plastic state were given in two papers by W. Prager.[10] These are not presented as absolute minima so that strict inequalities are not given, however, in each case it is verified that the first variation vanishes and that the second variation is positive. An absolute minimum principle for the stress rates for a general flow theory of a strain hardening material has recently been given by P. Hodge and W. Prager.[11] A *maximum* principle for the stresses for a theory of flow pertaining to a plastic material of the St. Venant-Lévy-Mises type has been given by R. Hill[12] under the assumption that the body is everywhere in the plastic state.

[9]W. Prager and J. L. Synge, *Approximations in elasticity based on the concept of function space*, Quart. Appl. Math. 5, 241-269, especially 249, (1947).

[10]See footnote 3 and the more general, *Variational principles in the theory of plasticity*, to appear in the Proc. 6th Internat. Congr. Appl. Mech., Paris, 1946.

[11]P. Hodge and W. Prager, *A variational principle for plastic materials with strain hardening*, to appear in J. Math. Phys.

[12]R. Hill, *A variational principle of maximum plastic work in classical plasticity*, to appear in the Quart. J. Mech. Appl. Math.

The principle of A. Haar and Th. v. Kármán[13] bears the same relationship to the deformation theory of H. Hencky[14] as the present principle of minimum stress rate intensity does to the flow theory of Prandtl and Reuss. Assuming the Haar-Kármán principle, Hencky obtained his stress-strain relations as the Euler-Lagrange equations for the integral being minimized. The Prandtl-Reuss equations can be similarly derived from the present minimum principle. Conversely, if one assumes the Hencky relations, the Haar-Kármán principle follows (this will be proved in a later note) just as we have shown the principle of minimum stress rate intensity to follow from the Prandtl-Reuss relations.

[13]A. Haar and Th. v. Kármán, *Zur Theorie der Spannungszustände in plastischen und sandartigen Medien*, Nachr. Ges. Wiss. Göttingen, 204-218 (1909).

[14]H. Hencky, *Über einige statische bestimmte Fälle des Gleichgewichts in plastischen Körpern*, Z. angew. Math. Mech., 241-251 (1923).

—NOTES—

A NOTE ON THE VIBRATING STRING*

By G. F. CARRIER (*Brown University*)

In a previous paper [1] certain characteristics of the behavior of an elastic string undergoing periodic vibrations of moderately large amplitude were established. In this note, the same basic result is obtained in a manner which is mathematically more satisfactory. Furthermore, a refinement of the solution associated with the periodic motion of lowest frequency is obtained.

It is convenient to postulate a material which obeys the stress-strain law

$$T - T_0 = EA\{[(1 + v_x)^2 + u_x^2]^{1/2} - 1\}. \tag{1}$$

Here, T_0 is the rest tension, A the rest cross-sectional area, E an elastic constant of the material, and T, u, v, θ, are defined[1] in Fig. 1. This law is probably as close to reality as any we could postulate for the general run of elastic materials. In any event, a modified Eq. (1) introduces into the results only higher order effects.

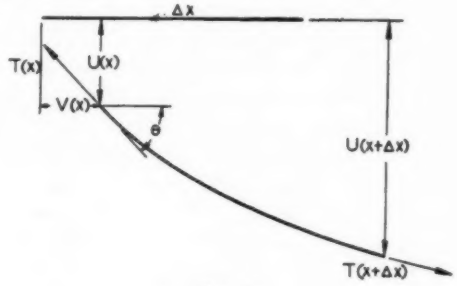

FIG. 1.

If we apply the conditions of dynamic equilibrium to an element of the string as shown in Fig. 1, we obtain (rigorously)

$$[T \sin \theta]_x = \rho A u_{tt} , \tag{2}$$

$$[T \cos \theta]_x = \rho A v_{tt} . \tag{3}$$

Now with $\tau = (T - T_0)/T_0$, $\xi = \pi x/l$, $\eta^2 = \pi^2\alpha^2 Et^2/\rho l^2$, $\alpha^2 = T_0/EA$, Eqs. (1), (2) and (3) can be combined to give

$$[(1 + \tau)e^{i\theta}]_{\xi\xi} = [(1 + \alpha^2\tau)e^{i\theta}]_{\eta\eta} . \tag{4}$$

The boundary conditions $u = 0$ and $v = 0$ at $\xi = 0$ and $\xi = \pi$ can be replaced by

$$\int_0^\pi (1 + \alpha^2\tau)e^{i\theta}\, d\xi = \pi. \tag{5}$$

These are essentially the basic equations used in [1].

*Received Feb. 25, 1948.

[1] We note that $\tan \theta = u_x/(1 + v_x)$.

In order to obtain those solutions of Eqs. (4) and (5) in which we are interested, it is convenient to define $\theta = \alpha w$, carry out the differentiations indicated in Eq. (4), factor the exponential, and separate the real and imaginary parts. We obtain

$$(1 + \tau)w_{\xi\xi} - w_{\eta\eta} = -2w_\xi\tau_\xi + \alpha^2(2w_\eta\tau_\eta + \tau w_{\eta\eta}), \tag{6}$$

$$\tau_{\xi\xi} = \alpha^2[\tau_{\eta\eta} + (1 + \tau)w_\xi^2 - w_\eta^2] - \alpha^4\tau w_\eta^2 . \tag{7}$$

Furthermore, Eq. (5) may be expanded to give

$$\int_0^\pi [(\tau - w^2/2) + \alpha^2(w^4/4! - w^2\tau/2) + \cdots] \, d\xi = 0. \tag{8}$$

Here, it is implied that we consider only functions w which are odd in ξ about the point $\pi/2$ and functions τ which are even about this point.

At this stage of the development, it is interesting to observe the analogy presented by this problem and the boundary layer problems in fluid dynamics [2]. In the boundary layer problem one replaces the coordinate normal to the boundary by a new coordinate which includes a natural parameter (i.e. the Reynold's number[2]) just as here α is incorporated in η. The velocity components of the fluid problem are of zero order and first order in α_1 , whereas in our problem, τ and θ are respectively of these orders. Finally, in the fluid problem, once the foregoing items are incorporated into the equations, one solves the equations for the case $\alpha_1 = 0$ and anticipates that this asymptotic solution will represent the physical picture very well when $\alpha_1 \ll 1$. This, of course, will also be the procedure here.

Returning to the problem then, we can deduce consecutively from Eqs. (7), (8), (6), that, for $\alpha = 0$,

$$\tau = \tau(\eta), \tag{9}$$

$$\tau = \int_0^\pi (w^2/2\pi) \, d\xi, \tag{10}$$

$$\left[1 + \int_0^\pi (w^2/2\pi) \, d\xi\right] w_{\xi\xi} - w_{\eta\eta} = 0. \tag{11}$$

Equation (11) corresponds to Eq. (17) of [1]. It leads directly to the solution for the low frequency periodic oscillation (as given in [1])

$$w = 2\epsilon u_0(\eta) \cos \xi \tag{12}$$

and

$$\tau = \epsilon^2 u_0^2(\eta), \tag{13}$$

where

$$u_0(\eta) = \mathrm{cn}[(1 + \epsilon^2)^{1/2}\eta, k_0] \tag{14}$$

and

$$k_0 = \epsilon/[2(1 + \epsilon^2)]^{1/2}. \tag{15}$$

[2]We shall call the Reynold's number α_1^{-2} here.

This solution can be expected to be a good approximation to the rigorous result for small α. It is interesting nevertheless to investigate the periodic solution which is valid to terms of order α^4. The procedure is necessarily somewhat unorthodox. It is convenient to write

$$w = 2\epsilon\varphi(\eta)\cos\xi + \alpha^2 \sum_{j=3,5,7}^{\infty} u_j(\eta)\cos j\xi, \tag{16}$$

$$\tau = \epsilon^2\varphi^2(\eta) + \alpha^2\tau_1(\xi, \eta) = \tau_0 + \alpha^2\tau_1 . \tag{17}$$

If these forms of w and τ are substituted into Eqs. (6), (7), (8), we obtain from Eq. (7) (discarding terms of order $\alpha^4, \cdots$)

$$(\tau_1)_{\xi\xi} = (\tau_0)_{\eta\eta} + 4\epsilon^2(1 + \tau_0)\varphi^2\sin^2\xi - 4\epsilon^2\cos^2\xi(\varphi_\eta)^2. \tag{18}$$

Using Eq. (8) to evaluate the arbitrary function of η which arises when we integrate Eq. (18), we obtain for τ_1

$$\begin{aligned}\tau_1 = {} & (\tau_0)_{\eta\eta}[(\xi - \pi/2)^2/2 - \pi^2/24] + \epsilon^2(1 + \tau_0)\varphi^2[(\xi - \pi/2)^2 \\ & - \pi^2/12 + (\cos 2\xi)/2] - \epsilon^2\varphi_\eta^2[(\xi - \pi/2)^2 - \pi^2/12 \\ & - (\cos 2\xi)/2] + \frac{3}{4}\epsilon^4\varphi^4.\end{aligned} \tag{19}$$

We may now expand the functions of ξ which occur in Eq. (19) in Fourier series containing only terms of the form $\cos 2n\xi$. Using these, the terms of order α^2 in Eq. (6) may be evaluated. In fact, we can write them so that they consist of series of the form $F_n\cos(2n + 1)\xi$, where the F_n are combinations of φ, τ_0, and their derivatives. When this is done we may combine all terms and equate the coefficient of $\cos(2n + 1)\xi$ to zero for each n. The equation associated with $n = 0$ is that which defines φ and (when terms in $\alpha^4, \cdots$, are discarded) is

$$\varphi_{\eta\eta} + \varphi + \epsilon^2\varphi^3 + \alpha^2[13\epsilon^2\varphi\varphi_\eta^2 - 2\epsilon^2\varphi^2\varphi_{\eta\eta} - 9\epsilon^2\varphi^3 - 6\epsilon^4\varphi^5]/4 = 0. \tag{20}$$

This equation is not easy to integrate but a simple substitution renders it tractable. For moderately small α one expects that φ will closely resemble u_0. Let us observe then, that

$$(u_0)_\eta^2 = (1 + \epsilon^2/2) - u_0^2 - \epsilon^2u_0^4/2 \tag{21}$$

and

$$(u_0)_{\eta\eta} = -u_0 - \epsilon^2u_0^3 , \tag{22}$$

and let us assume (or rather hope) that

$$\varphi_\eta^2 = 2(1 + \epsilon^2/2) - \varphi^2 - \epsilon^2\varphi^4/2 + O(\alpha^2), \tag{23}$$

$$\varphi_{\eta\eta} = -\varphi - \epsilon^2\varphi^3 + O(\alpha^2). \tag{24}$$

If these relations were true, then we could replace φ_η^2 and $\varphi_{\eta\eta}$ *in the coefficient of* α^2 in Eq. (20) and obtain an equation which is still accurate to order α^4. Since Eq. (20) is already in error to terms in α^4, this is a consistent procedure. Let us make this substitution then, with the reservation that we must substitute the solution obtained under

this assumption into Eq. (20) and verify that the terms "left over" are of order α^4. One can readily verify that the forthcoming answer satisfies this condition. When Eq. (20) is modified in this manner, it becomes

$$\varphi_{\eta\eta} + a_1\varphi + a_3\varphi^3 + a_5\varphi^5 = 0, \tag{25}$$

where

$$a_1 = 1 + 13\alpha^2\epsilon^2(2 + \epsilon^2)/8,$$

$$a_3 = \epsilon^2(1 - 5\alpha^2),$$

$$a_5 = -21\epsilon^4\alpha^2/8,$$

and the solution such $\varphi(0) = 1$, $\varphi'(0) = 0$, is[3]

$$\varphi = \frac{\operatorname{cn}(\beta\eta, k)}{1 - A\operatorname{sn}^2(\beta\eta, k)}, \tag{26}$$

where

$$\beta^2 = 1 + \epsilon^2 + \frac{3}{4}\alpha^2\epsilon^4,$$

$$k^2 = [\epsilon^2/2(1 + \epsilon^2)][1 - 5\alpha^2 - 7\alpha^2\epsilon^2/4 - 3\alpha^2\epsilon^4/4(1 + \epsilon^2)],$$

$$A = 7\alpha^2\epsilon^2/8,$$

when terms of order α^4 are discarded.

Substitution of this result into Eq. (20) will reveal that the equality fails only in terms of order α^4, and thus the desired result has been obtained. It is evident that for small $\alpha^2(\alpha^2 = .002$ is large for the usual elastic medium) the frequency and wave form differ very little from those of the first order solution.

There still remains the problem of finding u_3, u_5, $\cdots$. Again, substitution of Eq. (16) and (20) into Eq. (6) leads to the equation for u_3:

$$(u_3)_{\eta\eta} + 9(1 + \epsilon^2\varphi^2)u_3 = G(\eta). \tag{27}$$

Here, $G(\eta)$ is a group of terms of the type $(\tau_0)_{\eta\eta}$, φ, φ^3, $\cdots$. That is, $G(\eta)$ has the same period as φ.

It is known that the homogeneous solutions associated with Eq. (27) are of the type

$$u_3 = e^{\pm\lambda\eta}f(\eta),$$

where $f(\eta)$ has the period of φ^2. *If* λ *is purely imaginary, such solutions are stable and the natural frequencies are different from that of* φ^2. In such cases, the non-homogeneous solution of Eq. (27) can rarely exhibit a resonant effect. Since φ^2 does not differ greatly from a trigonometric function, the results obtained by Lubkin & Stoker [3] can be used to estimate whether stability of the homogeneous solution of Eq. (27) is implied. It is evident by inspection of their results that the solution is stable when ϵ is not large compared to unity. Therefore, we can conclude that the functions u_3, u_5, $\cdots$, are bounded and that we are dealing with a periodic solution of Eq. (6). The difficulties

[3]Elementary integration and the use of Pierce's integral tables are the only processes required here.

in actually obtaining u_3 , u_5 , $\cdots$, etc. are so great that we do not feel justified in pursuing this point here.

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TRANSONIC DRAG OF AN ACCELERATED BODY*

BY M. A. BIOT (*Brown University*)

It is known from the linear theory that the steady state drag of a body at the speed of sound is infinite. The occurrence of this infinite value may be interpreted as due to a resonance phenomenon and the accumulation of disturbances over an infinite interval of time. In non-steady motion, however, this resonance does not occur, and a finite value must be expected for the drag, which becomes smaller as the acceleration increases. The investigation of this phenomenon is the object of the present paper. An investigation of the drag of an accelerated body was made by F. J. Frankl.[1] His method however is approximate and does not apply at the speed of sound.

We consider a two-dimensional symmetric wedge of vertex angle 2α moving along the x-axis. The wedge is uniformly accelerated with an acceleration γ. The coordinate of the vertex O as a function of time t is (Fig. 1).

$$x = 1/2\,\gamma t^2. \tag{1}$$

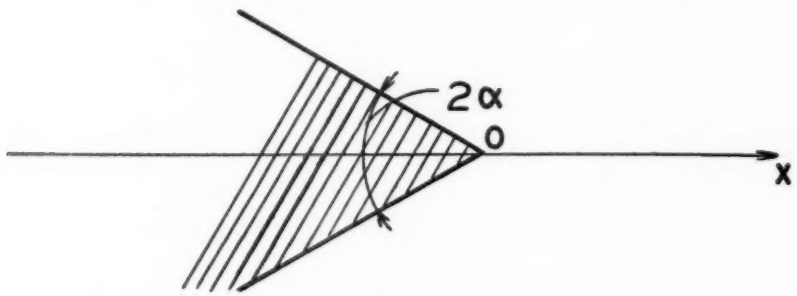

FIG. 1.

We shall simulate the motion of the solid wedge by distributing variable sources along the x-axis in such a way that the velocity component normal to x is the same as that

*Received Dec. 13, 1948. Presented at the 7th International Congress of Applied Mechanics, London, 1948.

[1] F. J. Frankl, *Influence of the acceleration of oblong bodies of revolution upon the resistance of the gas*, Inst. of Mech., Acad. Sci. USSR Appl. Math. Mech., Vol. X, 1946.

due to the wedge motion. We assume that the wedge angle is small and introduce the usual assumption of the linearized perturbation theory. The velocity potential φ of the perturbation satisfies the acoustic equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} \tag{2}$$

where c is the sound velocity. The basic solution from which all following expressions are derived is

$$\varphi = \frac{v}{\pi}\log\left[\frac{ct}{r} + \frac{(c^2t^2 - r^2)^{1/2}}{r}\right], \tag{3}$$

$$r = (x^2 + y^2)^{1/2},$$

which represents a source appearing suddenly at the origin at $t = 0$. The velocity field becomes identical with that of a source in an incompressible fluid for sufficiently small values of r. Therefore, if we distribute sources along the x-axis with intensities per unit length corresponding to expressions (3), the normal velocity generated by these sources will be a constant v appearing suddenly at $t = 0$. What we are interested in is the pressure associated with the potential φ. This pressure is

$$p = \rho\frac{\partial\varphi}{\partial t} = \frac{\rho vc}{\pi}\frac{1}{(c^2t^2 - r^2)^{1/2}}. \tag{4}$$

This may be generalized to cover the case of the accelerated wedge. The normal velocity at any point along x due to the wedge motion acquires a sudden value v_o when the nose of the wedge hits that point, and then continues to increase linearly following $v = v_o + \alpha\gamma t$. The pressure field of such a source is derived from (4) by superposition. Assuming this source to be located at the origin and starting at $t = 0$ the pressure is

$$p = \frac{\rho v_o c}{\pi}\frac{1}{(c^2t^2 - r^2)^{1/2}} + \frac{\rho c}{\pi}\int_0^t \frac{1}{[c^2(t - \tau)^2 - r^2]^{1/2}}\frac{dv}{d\tau}\,d\tau \tag{5}$$

with $dv/d\tau = \alpha\gamma$.

By integration this may be written

$$p = \frac{\rho v_o c}{\pi}\frac{1}{(c^2t^2 - r^2)^{1/2}} + \frac{\rho\alpha\gamma}{\pi}\log\left[\frac{ct}{r} + \frac{(c^2t^2 - r^2)^{1/2}}{r}\right]. \tag{6}$$

Actually this source is generated by the wedge at point ξ. We must distinguish between positive and negative values of ξ. For $\xi < 0$ all sources originate at an instant $t_1 = 0$ and the initial normal velocity is $v_o = 0$. For $\xi > 0$ the sources originate at the instant $t_1 = (2\xi/\gamma)^{1/2}$ when the nose of the wedge reaches that point and the initial normal velocity is $v_o = \gamma t_1 = (2\xi\gamma)^{1/2}$. With a step function $1(\xi)$

$$1(\xi) = 0 \text{ for } \xi < 0,$$

$$1(\xi) = 1 \text{ for } \xi > 0,$$

we may write the general expressions

$$t_1 = \left(\frac{2\xi}{\gamma}\right)^{1/2} 1(\xi),$$

$$v_0 = (2\xi\gamma)^{1/2} 1(\xi),$$

valid for the complete ξ-axis.

This particular source produces at point x and time t a pressure

$$p(x, t, \xi) = \frac{\rho c}{\pi} \frac{(2\xi\gamma)^{1/2} 1(\xi)}{\{c^2[t - (2\xi/\gamma)^{1/2} 1(\xi)]^2 - (x - \xi)^2\}^{1/2}} + \frac{\rho\alpha\gamma}{\pi} \log \frac{c[t - (2\xi/\gamma)^{1/2} 1(\xi)] + \{c^2[t - (2\xi/\gamma)^{1/2} 1(\xi)]^2 - (x - \xi)^2\}^{1/2}}{| x - \xi |}. \tag{7}$$

In order to obtain the pressure due to the motion of the wedge we must remember that expression (7) represents the pressure due to a source per unit length at point ξ. Hence the total pressure is obtained by multiplying expression (7) by $d\xi$ and integrating with respect to ξ.

$$p(x, t) = \int_{-\infty}^{+\infty} p(x, t, \xi)\, d\xi. \tag{8}$$

This represents the pressure distribution along the x-axis at any time t. In performing this integration we must remember that the integrand is to vanish whenever the radical becomes imaginary or whenever $t < (2\xi/\gamma)^{1/2} 1(\xi)$. The range of integration therefore only covers intervals of ξ such that

$$c^2\left[t - \left(\frac{2\xi}{\gamma}\right)^{1/2} 1(\xi)\right]^2 - (x - \xi)^2 \geq 0$$

and

$$t - \left(\frac{2\xi}{\gamma}\right)^{1/2} 1(\xi) \geq 0. \tag{9}$$

Consider the (x, ct) plane.
The parametric equation

$$ct = c\left(\frac{2\xi}{\gamma}\right)^{1/2} 1(\xi), \tag{10}$$

$$x = \xi$$

represents a curve AOB (Fig. 2) constituted by a straight line and a parabola.

Now consider the two straight lines of slope ± 1 which issue downward from a point P with coordinates (x, ct). They intersect the line AOB at points ξ_1, ξ_2, ξ_3, ξ_4. It may be verified that for this point P the expressions (9) are positive only for $\xi_1 < \xi < \xi_2$ and $\xi_3 < \xi < \xi_4$. Hence these intervals constitute the intervals of integration for (8). Depending on the location of P there may be two or four points of intersection.

The problem of determining the pressure distribution along the wedge at any particular instant t is thus solved.

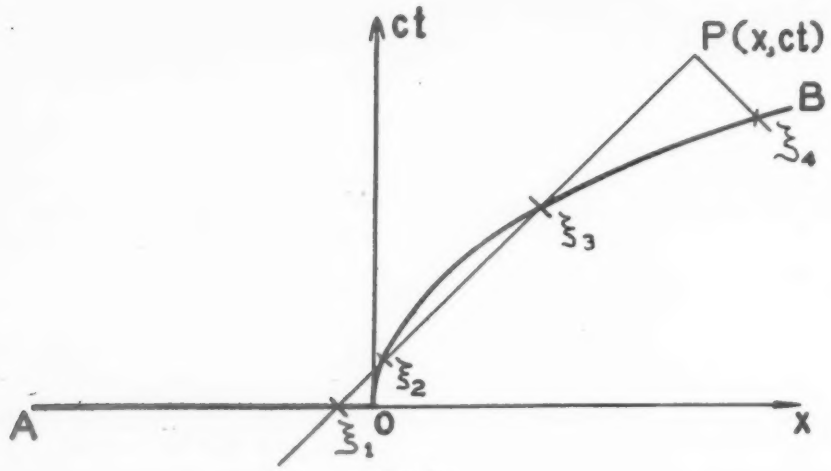

FIG. 2.

We shall not write explicitly the integrals in the general case but will focus our attention on the pressure distribution when the wedge reaches the speed of sound. In that case points ξ_2 and ξ_3 disappear. Expression (8) becomes

$$p(x) = p_1 + p_2 + p_3 \tag{11}$$

$$p_1 = \frac{\alpha\rho c^2}{2\pi} \int_{a-2}^{0} d\lambda \log \frac{2 + [4 - (a - \lambda)^2]^{1/2}}{| a - \lambda |},$$

$$p_2 = \frac{\alpha\rho c^2}{\pi} \int_{0}^{[-1+(3+a)^{1/2}]^2} \frac{\lambda^{1/2}\, d\lambda}{\{(2 - 2\lambda^{1/2})^2 - (a - \lambda)^2\}^{1/2}},$$

$$p_3 = \frac{\alpha\rho c^2}{2\pi} \int_{0}^{[-1+(3+a)^{1/2}]^2} d\lambda \log \frac{2 - 2\lambda^{1/2} + \{(2 - 2\lambda^{1/2})^2 - (a - \lambda)^2\}^{1/2}}{| a - \lambda |},$$

where $x/x_1 = a$ and $\xi/x_1 = \lambda$ are non-dimensional coordinates and $x_1 = c^2/2\gamma$ is the distance traveled by the wedge when it reaches the speed of sound. It is also assumed here that the coordinate $x > -ct$.

The value $a = 1$ corresponds to the nose. Note that for this value of a, p_1 and p_3 are finite, while p_2 becomes infinite. Hence p_2 is the preponderant contribution to the pressure for points near the nose. Because of the scale factor x_1, this is also equivalent to saying that p_2 is preponderant when the acceleration is small. We shall, therefore, assume that p_2 represents the pressure and evaluate the corresponding integrals in the vicinity of $a = 1$. By the change of variable

$$1 - \lambda^{1/2} = z(1 - a)^{1/2}$$

and for small values of $(1 - a)$ it is possible to show that the integral for p_2 tends toward the value

$$p_2 = \frac{4\alpha\rho c^2}{\pi(1 - a)^{1/4}} \int_0^{\infty} \frac{dz}{[(z^2 + 1)z]^{1/2}} = \frac{4.78}{(1 - a)^{1/4}} \alpha\rho c^2. \tag{12}$$

If we denote by x_2 the distance from the nose,

$$p_2 = 9.56 \frac{\alpha\rho c^2}{2} (x_1/x_2)^{1/4}. \tag{13}$$

The drag for a length l is

$$D = 2\alpha \int_0^l p_2 \, dx_2 = 25.5\alpha^2 (x_1/l)^{1/4} \frac{\rho c^2}{2} l. \tag{14}$$

The drag coefficient is

$$c_D = 25.5\alpha^2 (x_1/l)^{1/4} = 25.5\alpha^2 (c^2/2\gamma l)^{1/4}. \tag{15}$$

The drag depends on the ratio of the length of the body to the distance traveled from rest to reach the speed of sound. The presence of an acceleration causes a finite drag at the speed of sound in contrast to the infinite value in the steady case. As the value of the acceleration decreases the drag tends to infinity as the inverse fourth root of the acceleration. Another difference with the steady case is the concentration of infinite pressure at the nose. In this connection it may be concluded that the lift distribution on an accelerated wing will introduce a stalling moment in going through the speed of sound. It may be seen from formula (15) that extremely high values of the acceleration are needed for usual body sizes before the effect becomes appreciable.

It must be added that the methods presented in this paper are not restricted to the acceleration of a wedge. By superposition of positive and negative wedges the method solves the problem for a symmetric body of arbitrary shapes with constant acceleration. Furthermore, it will be noted that expressions (6), (7) and (8) may easily be generalized to cover not only the symmetric body of arbitrary shape but also the case of completely arbitrary motion. The present paper indicates how the pressure distribution in such cases may be completely expressed by quadratures.

A GENERALIZATION OF THE WIENER-HOPF TECHNIQUE*

By G. F. CARRIER (*Brown University*)

1. Introduction. Many of the problems of mathematical physics require the solution of an integral equation of the type

$$u(x) = \varphi(x) + \int_a^\infty K(x, x_0) f(x_0) \, dx_0 \, ,$$

where $u(x) = 0$ when $x > a$, and $f(x) = 0$ when $x < a$. When $K(x, x_0) = K(x - x_0)$ the equation is a Wiener-Hopf integral equation and the technique by which $f(x)$ may be found is well-known (cf. [1]). However, in many of the problems which arise K is not a function of $(x - x_0)$ and thus it seems desirable to generalize the Wiener-Hopf technique to include a more general family of kernels. In this paper, we shall concern ourselves with kernels which arise as the Green's functions of a certain family of partial differential equations. Although we shall choose as a basic problem a certain boundary

*Received Aug. 18, 1948.

value problem associated with such differential equations, it will be evident that the technique will be of use in questions which do not arise in this manner.

Specifically we shall consider the solutions of

$$\varphi_{yy} + y^m(\varphi_{xx} + k^2\varphi) = 0$$

which are composed of an "incident simple wave"[1] and a diffracted wave such that $\varphi(y, 0)$ vanishes when $y > b > 0$. We shall then include general remarks concerning other interesting cases.

2. The Basic Problem. Let us consider the boundary value problem associated with the equation

$$\varphi_{yy} + y^m(\varphi_{xx} + k^2\varphi) = 0 \tag{1}$$

where m is an integer greater than zero. It is required here that $\varphi(y, x)$ consist of a given "incident wave" $\varphi_0(y, x)$ plus a "diffracted wave", which together obey the requirement $\varphi(y, 0) \equiv 0$ when $y > b > 0$, and that $\varphi(y, x)$ be differentiable throughout $y > 0$ except on the cut mentioned above. In the region $y < 0$, we shall be content to accept the continuation of any solution which satisfies the foregoing conditions.

A natural way in which to formulate this problem is to write

$$\varphi = \varphi_0 + \int_a^\infty K(x, 0, y, y_0)f(y_0)\, dy_0 \tag{2}$$

where $K(x, x_0, y, y_0)$ is a Green's function[2] associated with Eq. (1). We must, then, find the Green's function, and find the function $f(y)$ associated with a given φ_0. To do these things we shall need certain transform identities which are quite simple and which, in fact, can be constructed as linear combinations of identities in conventional Hankel transforms. The essential facts are as follows. Let the H-transform of $f(s)$ be defined to be

$$\bar{f}(z) = \int_0^\infty z^\nu s^{1-\nu} H_\nu^{(2)}(sz) f(s)\, ds \tag{3}$$

Then

$$f(p) = \frac{1}{4}\int_{-\infty}^\infty z^{1-\nu} p^\nu H_\nu^{(1)}(pz)\bar{f}(z)\, dz \tag{4}$$

where the integration path is to pass above the origin and where negative z has argument $+\pi$. In particular, we shall need to know that the function $f_1(p)$ associated with $\bar{f}_1(z) = z^\nu/(z - \gamma)$ is (via Eq. (4)) $f_1(p) = \gamma p^\nu H_\nu^{(1)}(p\gamma)$.

In a previous paper [2], conventional transform methods have been used to find the Green's function associated with an equation similar to Eq. (1). Without giving the details, it may be stated that the same techniques (using the transform pairs given above) will give the Green's function of Eq. (1) as

[1] This simple wave can be thought of as the analog to the plane wave associated with the equation $\Delta\varphi + k^2\varphi = 0$.

[2] Actually there are two linearly independent Green's functions which vanish as $y \to \pm\infty$, $x \to \infty$ and are outgoing, but which differ in their boundary conditions along $y = 0$. We shall obviously use the convenient one.

$K(x, x_0, y, y_0) =$

$$\frac{1}{4}\int_{-\infty}^{\infty} y^{1/2} y_0^{\alpha-1/2} \frac{\eta H_\nu^{(1)}(2\nu\eta y^\alpha) H_\nu^{(2)}(2\nu\eta y_0^\alpha) \exp\{-|x - x_0|(\eta^2 - k^2)^{1/2}\}}{(\eta^2 - k^2)^{1/2}} d\eta. \tag{5}$$

where $\nu = (2 + m)^{-1}$, $\alpha = 1 + m/2$. For the case $k = 0$, this reduces to a Legendre function of the second kind, but for $k \neq 0$, it is not easy to evaluate.

The choice of φ_0 will not be arbitrary, of course, in an explicit physical problem; but here, where the problem is merely to exhibit the technique, we may choose φ_0 with an eye to simplifying the manipulations. Accordingly, we choose a function whose H-transform is simple. Such a function [which is a solution of Eq. (1)] is

$$\psi_0(y, x) = \left(\frac{k \cos\beta}{4}\right) y^{1/2} H_\nu^{(1)}(2\nu y^\alpha k \cos\beta) \exp\{ikx \sin\beta\}. \tag{6}$$

This choice is not as artificial as it may seem at first glance. Actually, almost any φ_0 which one would wish to associate with Eq. (2) can be built up of a linear combination of those above by writing

$$\varphi_0 = \int \psi_0(y, x) h(\beta)\, d\beta. \tag{7}$$

In any event, we shall use ψ_0 in the present problem.

It is convenient to define $s = 2\nu y^\alpha$ so that

$$\psi_0 = \left(\frac{k \cos\beta}{4}\right) s^\nu H_\nu^{(1)}(ks \cos\beta) \exp\{ikx \sin\beta\}, \tag{8}$$

$$K = \frac{1}{4}\int_{-\infty}^{\infty} s^\nu s_0^{1-\nu} \frac{\eta H_\nu^{(1)}(\eta s) H_\nu^{(2)}(\eta s_0) \exp\{-|x - x_0|(\eta^2 - k^2)^{1/2}\}}{(\eta^2 - k^2)^{1/2}} d\eta. \tag{9}$$

Thus Eq. (2) can be written

$$\varphi = \psi_0 + \int_a^\infty K g(s_0)\, ds_0 . \tag{10}$$

If we now let $x = 0$, the integral equation has its final form and we are ready to apply the generalized Wiener-Hopf technique. We operate on Eq. (10) as in Eq. (3) to obtain (recalling that the boundary condition is applied at $x = 0$)

$$\bar{\varphi}(z)\Big]_{x=0} = \frac{z^\nu}{z - k\cos\beta} + \frac{\bar{g}(z)}{(z^2 - k^2)^{1/2}}. \tag{11}$$

It is now convenient to define $z = \zeta^{1/\nu}$ and to substitute for z in Eq. (11). When this is done and when we let[3] $\epsilon = \text{Im } k > 0$, we can observe the following facts:

First:

$$\bar{g}(\zeta^{1/\nu}) = \int_a^\infty \zeta s^{1-\nu} H_\nu^{(2)}(s\zeta^{1/\nu}) g(s)\, ds$$

is a function of ζ which is analytic throughout[4] the region R, which contains the sectors

[3] When the answer has been obtained, we can, of course, let $\epsilon \to 0$.

[4] This includes the implication that $\bar{g}$ is bounded as $|\zeta|$ tends to ∞ in R.

$(2n - 1)\pi \leq \arg \zeta^{1/\nu} \leq 2n\pi$. In fact, this statement is true for $\bar{g} \exp(ia\zeta^{1/\nu})$. This follows from the fact that $f(s) \sim e^{-s^2}$ as $s \to \infty$, and that $H_\nu^{(2)}(\alpha) \sim e^{-i\alpha}$ as $|\alpha| \to \infty$.

Second:

$$\exp(ia\zeta^{1/\nu})\bar{\varphi}(z)\Big]_{x=0} = \exp(ia\zeta^{1/\nu}) \int_0^a \zeta s^{1-\nu} H_\nu^{(2)}(s\zeta^{1/\nu})\varphi(y, 0)\, ds$$

is analytic in the region R' which includes the sectors $2n\pi \leq \arg \zeta^{1/\nu} \leq (2n + 1)\pi$. Thus, the regions of analyticity of these functions actually overlap at all the boundaries of R and R' (the regions are shown for $\nu = 1/3$ in Fig. 1) and hence cover the entire plane including the point at infinity.

Finally, multiplication of Eq. (11) by $\exp(ia\zeta^{1/\nu})\ (\zeta^{1/\nu} + k)^{1/2}$ yields (after a little algebra)

$$\bar{\varphi}(\zeta^{1/\nu} + k)^{1/2} \exp(ia\zeta^{1/\nu})$$

$$- \frac{\{(\zeta^{1/\nu} + k)^{1/2} \exp(ia\zeta^{1/\nu}) - [k(\cos\beta + 1)]^{1/2} \exp(iak \cos\beta)\}\zeta}{(\zeta^{1/\nu} - k \cos\beta)} \tag{12}$$

$$= \frac{\zeta[k(\cos\beta + 1)]^{1/2} \exp(iak \cos\beta)}{(\zeta^{1/\nu} - k \cos\beta)} + \frac{\bar{g}(\zeta^{1/\nu}) \exp(ia\zeta^{1/\nu})}{(\zeta^{1/\nu} - k)^{1/2}}.$$

The left side of this equation is regular in R' and the right side is regular in R. Thus, according to the arguments of analytic continuation, each of these functions is the con-

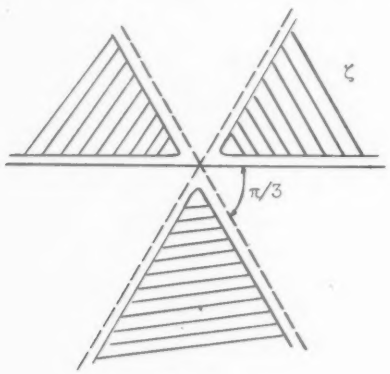

FIG. 1.

Qualitatively, the region R (that in which $\bar{g}(\zeta^3) \exp(ia\zeta^3)$ is analytic) is that excluding the shaded regions. R' is the region excluding these shaded regions after they have been rotated $\pi/3$ radians.

tinuation of the other and hence is analytic everywhere and therefore equal to a constant. Inspection of (12) together with footnote (4) immediately reveals that this constant is zero. Therefore,

$$\bar{g}(z) = \frac{[k(1 + \cos\beta)]^{1/2}(z - k)^{1/2} z^\nu \exp(-ia(z - k\cos\beta)}{(z - k\cos\beta)}, \tag{13}$$

and the desired result is given by Eq. (4). Thus, in principle, we have an answer. In a given physical problem, of course, one wishes to know the behavior of φ in certain regions. In many such cases conventional methods of interpreting the transforms will give asymptotic and local behaviors.

3. The General Problem. In this section some general but non-detailed remarks will be stated concerning the directions in which the foregoing may be readily generalized.

We have already implied that when φ_0 can be constructed as in Eq. (7), then the solution for φ can also be formulated as an integral over β of the solution we have just obtained. In the case where φ_0 has a simple transform (i.e. an easily factored transform), such an integration is obviously unnecessary.

A more general class of kernels which would be suitable in (2) can be constructed in the following way. Suppose there are a pair of functions $F_1(\xi, z)$, $F_2(\eta, z)$ such that

$$f(\xi) = \int_{-\infty}^{\infty} F_1(\xi, z) \int_a^b F_2(\eta, z) f(\eta)\, d\eta\, dz \tag{14}$$

for all f in $L^{(2)}$. Then, again in principle at least, the kernel

$$K = \int_{-\infty}^{\infty} F_1(s, z) F_2(s_0, z) h(z)\, dz \tag{15}$$

would be an appropriate one on which to use the foregoing technique, provided, of course, that the function F_2 has the sort of asymptotic behavior and the regularity which are needed if the foregoing analytic continuation arguments are to carry through. In Eq. (15), $h(z)$ is any function such that K exists and has a transform to which the arguments apply. From a practical point of view, h must produce a K whose transform can be factored in the manner used in the foregoing.

When $k = 0$, the present technique fails. However, if one solves the problem for $k \neq 0$ and then performs the limiting process $k \to 0$ (obvious in principle but messy in practice) solutions can be obtained.[5]

Finally, it seems evident that the homogeneous integral equation can also be treated with these more general transforms.

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[2] G. F. Carrier and F. E. Ehlers, *On some singular solutions of the Tricomi equation*, Quart. Appl. Math. **6**, 331–334 (1948).

[5] A. E. Heins has done this in the conventional Wiener-Hopf case.

VARIATIONAL PRINCIPLES IN THE THEORY OF FINITE PLASTIC DEFORMATIONS*

BY

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1. Introduction. In the present paper we give the variational principles of the theory of finite plastic deformations taking into account strain-hardening. Throughout the paper only Lagrangian coordinates will be used. This work is based on a paper of R. Kappus,[1] where a very clear exposition of the theory of finite elastic deformations is given.

The strain components are

$$g_{pq} = u_{q,p} + u_{p,q} + u_{i,p}u_{i,q} , \tag{1}$$

where u_p is the displacement and a subscript after a comma denotes differentiation with respect to the corresponding coordinate. The geometrical significance of these strain components is the following. Let us consider the system $(O, e_i^{(1)}, e_i^{(2)}, e_i^{(3)})$ of the particle O of the body and the three unit vectors $e_i^{(1)}$, $e_i^{(2)}$, $e_i^{(3)}$ which are parallel to the axes of the fixed coordinate system. The deformation brings O to O_1 , and $e_i^{(1)}$, $e_i^{(2)}$, $e_i^{(3)}$ to $g_i^{(1)}$, $g_i^{(2)}$, $g_i^{(3)}$, respectively.

The strain components g_{pq} are given by the formulas

$$g_{pq} = g_i^{(p)} g_i^{(q)} - \delta_{pq} , \tag{2}$$

where δ_{pq} is the Kronecker delta. They are components of a tensor.

Let us consider now the cubic element built on the vectors $e_i^{(1)}$, $e_i^{(2)}$, $e_i^{(3)}$. After deformation has taken place it became a parallelopiped built on the three vectors $g_i^{(1)}$, $g_i^{(2)}$, $g_i^{(3)}$. The forces on the faces of this parallelopiped are $\tau_i^{(1)}$, $\tau_i^{(2)}$, $\tau_i^{(3)}$.

Let us resolve them as follows:

$$\tau_i^{(p)} = \tau_{pq} g_i^{(q)} . \tag{3}$$

The components τ_{pq} are called the stress components. They are the components of a tensor and they fulfill the conditions

$$\tau_{pq} = \tau_{qp} , \tag{4}$$

The equations of equilibrium are

$$(\tau_{iq} + \tau_{pq} u_{i,p})_{,q} + F_i = 0 \tag{5}$$

and the boundary conditions are

$$(\tau_{iq} + \tau_{pq} u_{i,p}) \nu_q = F_i' , \tag{6}$$

where ν_q are the direction cosines of the exterior normal.

The stress-strain relations of the theory of finite elastic deformations are taken in the form

*Received May 10, 1948. The results presented in this paper were obtained in the course of research conducted under contract, sponsored by the Office of Naval Research.

[1] R. Kappus, Z. angew. Math. Mech. 19, 271 (1939).

$$\tau_{pq} = G\left(g_{pq} - \frac{1}{3}\, g_{ii}\delta_{pq}\right) + K\,\frac{1}{3}\, g_{ii}\delta_{pq}\,. \tag{7}$$

By putting

$$\tau^*_{pq} = \tau_{pq} - \frac{1}{3}\,\tau_{ii}\delta_{pq}\,, \qquad g^*_{pq} = g_{pq} - \frac{1}{3}\, g_{ii}\delta_{pq}\,,$$
$$T^* = (\tau^*_{pq}\tau^*_{pq})^{1/2}, \qquad D^* = (g^*_{pq}g^*_{pq})^{1/2}, \tag{8}$$

we get the two equations

$$\tau_{ii}/g_{ii} = k \tag{9}$$

$$T^* = GD^*. \tag{10}$$

The theory of finite plastic deformations assumes that Eq. (9) is valid again but that Eq. (10) will be replaced by

$$T^* = f(D^*) \tag{11}$$

i.e. T^* is a function of D^*.

Thus Eq. (7) becomes

$$\tau_{pq} = \frac{f(D^*)}{D^*}\, g^*_{pq} + K\,\frac{1}{3}\, g_{ii}\delta_{pq}\,. \tag{12}$$

Solving Eq. (11) for D^* we get

$$D^* = \phi(T^*)$$

and

$$g_{pq} = \frac{\phi(T^*)}{T^*}\,\tau^*_{pq} + \frac{1}{K}\,\frac{1}{3}\,\tau_{ii}\delta_{pq}\,. \tag{13}$$

2. The variational principle for the virtual displacements. Let us consider the function

$$A_0 = \int_0^{D^*} f(D^*)\, dD^* + \frac{3}{2}\, Kg^2, \tag{14}$$

where $g = g_{ii}/3$. We find

$$\frac{\partial A_0}{\partial g_{pq}} = f(D^*)\,\frac{\partial D^*}{\partial g_{pq}} + 3Kg\,\frac{\partial g}{\partial g_{pq}}$$
$$= \frac{f(D^*)}{D^*}\, g^*_{pq} + Kg\delta_{pq} = \tau_{pq}\,. \tag{15}$$

We consider now a body in equilibrium under the action of external forces. We denote by τ_{pq} the actual stress components, by u_i the components of the actual displacements and by g_{pq} the actual strain components, due to the loads.

We compare the actual value of the expression

$$P = \int_V A_0\, dV - 2\int_V F_i u_i\, dV - 2\int_S F'_i u_i\, dS \tag{16}$$

with the value which this expression would have if u_i and g_{pq} were submitted to small variations δu_i , δg_{pq} while the stress components and the external forces remained un-

changed. These variations are not entirely free; the new strain components $g_{pq} + \delta g_{pq}$ and displacements $u_i + \delta u_i$ must fulfill the following conditions:

a) the condition of continuity,
b) the geometrical boundary condition,
c) the equations of equilibrium

$$\{\tau_{iq} + \tau_{pq}(u_i + \delta u_i)_{,p}\}_{,q} + F_i = 0, \tag{17}$$

d) the statical boundary conditions

$$\{\tau_{iq} + \tau_{pq}(u_i + \delta u_i)_{,p}\}\nu_q = F_i' . \tag{18}$$

We shall prove that the actual value of P is a minimum. Let us write $A = \int_V A_0 dV$. We get

$$\delta A = \int_V \delta A_0 \, dV = \int_V \frac{\partial A_0}{\partial g_{pq}} \delta g_{pq} \, dV = \int_V \tau_{pq} \delta g_{pq} \, dV \tag{19}$$

Using the Eqs. (1) we obtain

$$\delta A = \int_V [\tau_{pq}(\delta u_{p,q} + \delta u_{q,p} + u_{i,p}\delta u_{i,q} + u_{i,q}\delta u_{i,p})] \, dV. \tag{20}$$

Integrating by parts and noting Eq. (4), we find that Eq. (20) becomes

$$\begin{aligned} \delta A = & - \int_V [\delta u_p \tau_{pq,q} + \delta u_q \tau_{pq,p} + \delta u_i(\tau_{pq} u_{i,p})_{,q} \\ & + u_i(\tau_{pq}\delta u_{i,q})_{,p} + \delta u_i(\tau_{pq} u_{i,q})_{,p} + u_i(\tau_{pq}\delta u_{i,p})_{,q}] \, dV \\ & + \int_S [\delta u_p \tau_{pq}\nu_q + \delta u_q \tau_{pq}\nu_p + \delta u_i(\tau_{pq} u_{i,p})\nu_q \\ & + u_i(\tau_{pq}\delta u_{i,q})\nu_p + \delta u_i(\tau_{pq} u_{i,q})\nu_p \\ & + u_i(\tau_{pq}\delta u_{i,p})\nu_q] \, dS. \end{aligned} \tag{21}$$

Using the Eqs. (5), (6) and (17), (18) we get

$$\delta A = 2 \int_V F_i \delta u_i \, dV + 2 \int_S F_i' \delta u_i \, dS, \tag{22}$$

and because F_i , F_i' remain constant during the variations,

$$\delta\left[\int_V A_0 \, dV - 2 \int_V F_i u_i \, dV - 2 \int_S F_i' u_i \, dS\right] = 0 \tag{23}$$

or

$$\delta P = 0. \tag{24}$$

This means that P is an extremum. We have to prove now that P is a minimum. We have

$$\delta^2 P = \delta^2 \int_V A_0 \, dV = \int_V \delta^2 A_0 \, dV. \tag{25}$$

But

$$\delta^2 A_0 = \frac{df(D^*)}{dD^*}(\delta D^*)^2 + f(D^*)\delta^2 D^*. \tag{26}$$

The first product of the right member of this equation is positive because for all materials $df(D^*)/dD^* > 0$. Let us examine the second product. We have

$$\delta^2 D^* = \frac{\partial^2 D^*}{\partial g^*_{pq}\partial g^*_{rs}}\,\delta g^*_{pq}\delta g^*_{rs}\,. \tag{27}$$

But

$$\frac{\partial^2 D^*}{\partial g^*_{pq}\partial g^*_{rs}} = \frac{D^*\delta^2_{pqrs} - g^*_{pq}g^*_{rs}}{D^{*3}}, \tag{28}$$

where

$$\delta_{pqrs} = \begin{cases} 1 & \text{for} \quad pq = rs, \\ 0 & \text{for} \quad pq \neq rs. \end{cases}$$

Hence

$$\begin{aligned} \delta^2 D^* &= \frac{D^{*2}(\delta g^*_{ij}\delta g^*_{ij}) - (g^*_{ij}\delta g^*_{ij})^2}{D^{*3}} \\ &= \frac{1}{D^{*3}}\left[(g^*_{pq}\delta g^*_{rs} - g^*_{rs}\delta g^*_{pq})(g^*_{pq}\delta g^*_{rs} - g^*_{rs}\delta g^*_{pq})\right] > 0. \end{aligned} \tag{29}$$

Therefore $\delta^2 A > 0$ and $\delta^2 P > 0$.

3. The variational principle for the virtual stress variations. Let us consider the function

$$B_0 = \int_0^{T^*} \phi(T^*)\,dT^* + \frac{3}{2}\frac{\tau^2}{K} \tag{30}$$

where $\tau = \tau_{ii}/3$. We have

$$\begin{aligned} \frac{\partial B_0}{\partial \tau_{pq}} &= \phi(T^*)\frac{\partial T^*}{\partial \tau_{pq}} + \frac{3}{K}\tau\frac{\partial \tau}{\partial \tau_{pq}} \\ &= \frac{\phi(T^*)}{T^*}(\tau_{pq} - \tau\delta_{pq}) + \frac{\tau}{K}\delta_{pq} = g_{pq}\,. \end{aligned} \tag{31}$$

Let us consider again a body in equilibrium under the action of external forces. We consider the expression

$$Q = \int_V B_0\,dV - 2\int_{S_\sigma} F'_i u_i\,dS \tag{32}$$

in which the second integral is extended over the part of the surface of the body where the boundary conditions are such that the external forces F'_i are not given. We compare the actual value of Q with the value which Q would have if the stress components

and the surface forces were submitted to small variations, while the strain components, the displacements, and the body forces remained unchanged. The only conditions which the variations must fulfill are that the new stress components and boundary forces must satisfy the equilibrium equations and the boundary conditions. We shall prove that the actual value of Q is a minimum.

Let us write

$$B = \int_V B_0 \, dV. \tag{33}$$

We get

$$\delta B = \int_V \delta B_0 \, dV = \int_V \frac{\partial B_0}{\partial \tau_{pq}} \, dV = \int_V g_{pq} \delta \tau_{pq} \, dV. \tag{34}$$

Using the Eqs. (1) we find that

$$\delta B = \int_V [\delta \tau_{pq}(u_{p,q} + u_{q,p} + u_{i,p} u_{i,q})] \, dV. \tag{35}$$

Integrating by parts we obtain

$$\begin{aligned} \delta B = & - \int_V [u_p \delta \tau_{pq,q} + u_q \delta \tau_{pq,p} + u_i (\delta \tau_{pq} u_{i,q})_{,p} \\ & + u_i (\delta \tau_{pq} u_{i,p})_{,q}] \, dV + \int_S [u_p \delta \tau_{pq} \nu_q \\ & + u_q \delta \tau_{pq} \nu_p + u_i (\delta \tau_{pq} u_{i,q}) \nu_p + u_i (\delta \tau_{pq} u_{i,p}) \nu_q] \, dS \end{aligned} \tag{36}$$

in which the second integral is valid for the whole boundary. The stress variations have been selected such that the equations of equilibrium are always satisfied. Hence the first integral vanishes.

In the part of the boundary where the external forces are given the second integral vanishes because the stress variations have been selected so that the boundary equations are satisfied. The second integral is different from zero in those parts of the boundary only where the external forces are not given. Then we have

$$\delta B = 2 \int_{S_0} u_i \delta F_i' \, dS, \tag{37}$$

where S_0 means the above-mentioned part of the boundary. As u_i remain constant during the variation, we get

$$\delta \left[\int_V B_0 \, dV - 2 \int_{S_0} u_i F_i' \, dS \right] = 0 \tag{38}$$

and

$$\delta Q = 0. \tag{39}$$

Hence Q is an extremum. The proof that Q is a minimum is similar to the proof for the minimum property of P.

The variational principles of the theory of infinitely small plastic deformations are special cases of the two principles given above. These special cases have been treated by L. M. Kachanov[2] and by the author[3] independently.

[2] L. M. Kachanov, Prikl. Mat. Mekh. **6**, 187-196 (1942).

[3] A. Philippidis, *Spannungen und Verformungen im überelastischen Bereich*, Münich, 1944. Unpublished Report.

A CRITERION OF OSCILLATORY STABILITY*

By AUREL WINTNER (*Johns Hopkins University*)

1. In the linear differential equation

$$x'' + f(t)x = 0, \tag{1}$$

let $f(t)$ be a real-valued, continuous function for large positive t, say for $t_0 \leqq t < \infty$. Consider only those solutions $x(t)$ of (1) which are real-valued and distinct from the trivial solution (identically zero). Then, if $N(t)$ denotes the number of zeros of $x(s)$ on the interval $t_0 \leqq s \leqq t$, it is clear from Sturm's separation theorem that

$$N(t) \to \infty \qquad \text{as} \qquad t \to \infty \tag{2}$$

cannot hold for a particular solution $x(t)$ unless it holds for every solution $x(t)$. In this case, (1) is called oscillatory. Thus (1) is called non-oscillatory if one (hence every) non-trivial solution fails to acquire an infinity of zeros, as $t \to \infty$.

In the applications, the importance of the classification of the differential equations (1) into the oscillatory and non-oscillatory categories is due to the following well-known fact: A non-trivial solution of (1) must change its sign whenever it vanishes, since $x(t)$ and $x'(t) = dx\ (t)/dt$ cannot vanish simultaneously.

A general criterion has been developed† for the type of stability defined by (2). In what follows, a sufficient criterion that $x(t)$ be oscillatory will be given which is not contained in known explicit tests.

2. It is easy to see that, if

$$f(t) \geqq 0, \tag{3}$$

(1) must be oscillatory whenever the indefinite integral

$$F(t) = \int^t f(s)\, ds \tag{4}$$

satisfies the condition

$$F(t) \to \infty \qquad \text{as} \qquad t \to \infty. \tag{5}$$

In fact, (3) and (1) imply that the graph of every solution $x = x(t)$ must always turn its concavities toward the t-axis of the (t, x)-plane. Hence, if (1) is non-oscillatory, and if the solution is so chosen that $x(t) > 0$ as $t \to \infty$, it is clear that $x(t) \geqq c$ holds for every sufficiently large t and for a positive constant c. Consequently, from (1) and (3),

$$x'(t) - \text{const.} = -\int^t x(s)f(s)\, ds \leqq -c\int^t f(s)\, ds.$$

Since $-c < 0$, it now follows from (4) and (5) that, if t is large enough, $x'(t)$ is negative, hence $x(t)$ is decreasing. But this contradicts the assumption that $x = x(t)$ is ultimately positive, and concave toward the t-axis, as $t \to \infty$.

By a substantial refinement of this argument, it will be shown below that, *if* (4) *satisfies* (5), *then* (1) *must be oscillatory, whether* (3) *is assumed or not.*

*Received July 6, 1948.

†A. Wintner, Quart. Appl. Math. 5, 232-236 (1947).

The usefulness of this criterion becomes clear if it is observed that the following fact results as a corollary: If $f(t)$ is periodic, or just almost-periodic, and if the mean-value of $f(t)$, that is, the constant term, a_0 , in the (harmonic or anharmonic) Fourier expansion,

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(\lambda_n t - \alpha_n), \qquad (\lambda_n \neq 0),$$

of $f(t)$ is positive, then (1) must be oscillatory. In fact, the function (4) is now asymptotically equal to $a_0 t$, and so (5) is satisfied if $a_0 > 0$.

3. Since (5) implies that

$$\left\{ \int^t F(s)\, ds \right\} / t \to \infty \qquad \text{as} \qquad t \to \infty, \tag{6}$$

more than the last italicized statement is contained in the following theorem:

Equation (1) *must be oscillatory if* (4) *satisfies* (6).

In order to prove this theorem, suppose that (1) is non-oscillatory. Then, if $x(t)$ is any real-valued, non-trivial solution of (1), and if t is large enough, it can be assumed that $x(t) > 0$ (in fact, if $x(t)$ is a solution, then $-x(t)$ is). Since $x(t)$ does not vanish, it has a logarithmic derivative which, in view of (1), satisfies Riccati's differential equation $q' = -f - q^2$, where $q = x'/x$. This differential equation implies the inequality $f \leqq -q'$. Hence, two quadratures and the definition (4) show that

$$\int^t F(s)\, ds \leqq - \int^t q(s)\, ds + at + b,$$

where a, b are integration constants. Since $q(t) = x'(t)/x(t)$ and $x(t) > 0$, it follows that, if b is suitably chosen,

$$\int^t F(s)\, ds - at - b \leqq - \log | x(t) |. \tag{7}$$

If (6) is assumed, then (7) implies that $\log | x(t) | \to -\infty$, i.e., that

$$x(t) \to 0 \qquad \text{as} \qquad t \to \infty. \tag{8}$$

Hence, if (1) is non-oscillatory, then (6) cannot hold unless every solution of (1) satisfies (8). Consequently, in order to complete the proof of the last italicized theorem, it will be sufficient to verify the following assertion: (1) must be oscillatory if every solution of (1) satisfies (8).

4. Since what is claimed by this assertion is just the truth of (2), more than what is needed is contained in the following fact:

Equation (8) *cannot hold for every solution of* (1) *unless*

$$N(t)/t \to \infty \qquad \text{as} \qquad t \to \infty. \tag{9}$$

The deduction of the refinement (9) of (2) proceeds as follows:

Let $x = x(t)$ and $x = y(t)$ be two linearly independent solutions of (1). Their Wronskian, being a non-vanishing constant, can be assumed to be 1. Then $r^2\phi' = 1$ is an identity in t, if $r = r(t) > 0$ and a (continuous) $\phi = \phi(t)$ are defined by placing $x = r\cos\phi$ and $y = r\sin\phi$.

Suppose that (8) holds for every solution of (1). Then $x(t) \to 0$ and $y(t) \to 0$ as $t \to \infty$. It follows therefore from $r^2(t)\phi'(t) = 1$, where $r^2 = x^2 + y^2$, that $\phi'(t) \to \infty$.

Hence, a quadrature shows that $\phi(t)/t \to \infty$, as $t \to \infty$. In view of $x(t) = r(t) \cos \phi(t)$, this implies the truth of (9) for the number of the zeros of $x(t)$.

It follows in the same way that

$$N(t)/t > \text{const.} > 0 \qquad \text{as} \qquad t \to \infty, \tag{10}$$

if every solution of (1), instead of satisfying (8), remains just bounded as $t \to \infty$. Similarly, *if* $x(t)/t^{1/2}$ *remains bounded for every solution of* (1), *then* (1) *must be oscillatory*, since (2) then follows in the same way as (9) and (10) did.

THE DISTRIBUTION OF PLANE ANGLES OF CONTACT*

By J. WOLFOWITZ (*Columbia University*)

Let $R = \omega(\theta)$ be a closed, simple, convex, plane curve, with a continuous tangent and the origin in its interior, and with R, θ its polar coordinates. A particle A moves in a straight line at constant speed k from time $t = 0$ to $t = T$. Another particle B moves with unit speed in a straight line coplanar with the path of A. The probability that at time $t = 0$ the particle B will be in any sub-region g of a sufficiently large region G containing the starting point of A (at time $t = 0$) is proportional to the area of g. The probability that the azimuth of B's motion will lie between the angles α, β, is $(\beta - \alpha)/2\pi$ and is independent of the starting position of B. Let θ be the angle from the direction of A's motion to the particle B, and R the distance from A to B. When $R = \omega(\theta)$ a "contact" is said to occur. Let ϕ be the angle from the radius vector from A to B to the direction of B's motion. The couple (θ, ϕ) characterize a contact. Naturally no contact need ever occur between $t = 0$ and $t = T$, or at any other time, for that matter. In this note we shall be concerned with the probability distribution of (θ, ϕ) in the totality of contacts which *do* occur between $t = 0$ and $t = T$, and excluding the possibility that, at $t = 0$, $R < \omega(\theta)$. We will show that the probability density $f(\theta, \phi)$ of (θ, ϕ) is $c(D^* + |D^*|)$ where

$$D^* = -\omega(\theta) \cos \phi + \frac{d\omega}{d\theta} \sin \phi + k\left[\omega(\theta) \cos \theta + \frac{d\omega}{d\theta} \sin \theta\right]$$

and c is a constant defined by

$$\int_0^{2\pi} \int_0^{2\pi} f(\theta, \phi)\, d\theta\, d\phi = 1.$$

The curve $R = \omega(\theta)$ may be thought of as a field of force surrounding the particle A. If A and B are circular disks and $\omega(\theta)$ identically equal to the sum of their radii, a "contact" would be literally such. The following should also be noted:

a) The result is independent of T. The requirement that G should be sufficiently large means that G should contain a circle, centered at the position of A at $t = 0$, and of radius sufficiently large (the minimum radius depends upon T). Once G is sufficiently large, $f(\theta, \phi)$ does not depend upon G.

b) The result depends only upon the ratio k of the speeds. The constant c depends

*Received July 13, 1948.

upon k, in general. If k is a chance variable distributed between k_1 and k_2 with probability density $m(k)$, the probability density of (θ, ϕ) is then

$$\int_{k_1}^{k_2} f(\theta, \phi) m(k)\, dk$$

(The minimum size of G then depends not only on T, but also on k_2).

c) The result is valid when A is at rest ($k = 0$).

d) The result holds without change if the azimuth and initial position of A are chance variables with any distributions whatever, provided only that they be independent of the initial position and azimuth of B.

e) When $\omega(\theta)$ is identically constant the parameter c may be evaluated without difficulty in terms of Legendre's elliptic integrals of the first and second kind.

f) When A is at rest and $\omega(\theta)$ is a constant, $f(\theta, \phi)$ is proportional to $\cos\phi$ when $\pi/2 < \phi < 3\pi/2$, and is zero otherwise.

g) The method of proof is applicable to the problem of contacts in space, except that the computations become much more involved.

We now proceed to the proof.

The direction of motion of particle A being $\theta = 0$, the direction of motion of particle B when its contact is characterized by θ, ϕ is $(\theta + \phi)$. Let x, y, $(\phi + \theta) = \psi$, be, respectively, Cartesian coordinates of position and the azimuth of the direction of motion of particle B at time $t = 0$. The origin of the Cartesian system will be placed at the position of A at $t = 0$ and the positive x-axis in the direction $\theta = 0$. Let H be the rectangle in θ, ϕ space whose vertices are (θ, ϕ), $(\theta + \Delta\theta, \phi)$, $(\theta, \phi + \Delta\phi)$ and $(\theta + \Delta\theta, \phi + \Delta\phi)$, respectively. Then the probability Π of a contact characterized by a point in this rectangle H is a constant multiplied by the volume of a region S in x, y, ψ space. This region contains those (and only those) points x, y, ψ which are such that, if the position and motion of particle B at time $t = 0$ are described by x, y, ψ, a contact in H will occur within $0 \leq t \leq T$. The desired probability density $f(\theta, \phi)$ is given by

$$f(\theta, \phi) = \lim_{\substack{\Delta\theta \to 0 \\ \Delta\phi \to 0}} \frac{\Pi}{\Delta\theta \Delta\phi}.$$

We shall now show that every section of S cut by a plane $\psi = \text{constant}$ is, to within infinitesimals of order higher than the first, a parallelogram. Let a contact θ, ϕ occur at time t, $0 \leq t \leq T$. Then at time t the position of A will be $(kt, 0)$, the position of B will be $(kt + R\cos\theta, R\sin\theta)$. The position of B at $t = 0$ was

$$(kt + R\cos\theta - t\cos\psi, R\sin\theta - t\sin\psi).$$

Hence, in order to have a contact (θ, ϕ) the particle B must be, at $t = 0$, on a line running from

$$M = (R\cos\theta, R\sin\theta)$$

to

$$M' = (R\cos\theta + kT - T\cos\psi, R\sin\theta - T\sin\psi).$$

Let now θ, ϕ move in the rectangle H along the line $\psi = \text{constant}$. Both M and M' are displaced by

$$\delta(-R\sin\theta + R'\cos\theta, R\cos\theta + R'\sin\theta) = (c, d)$$

where $\delta = \delta\theta = -\delta\phi$, and $\delta\theta$ and $\delta\phi$ are the increments of θ and ϕ, respectively. (Here $R' = d\omega(\theta)/d\theta$.) Thus the parallelogram has vertices

$$M,\ M',\ M + (c, d),\ M' + (c, d).$$

Its area, to within infinitesimals of order higher than the first, is $D\delta$, where D will be evaluated shortly. D is constant in H.

Let $\delta(\psi)$ be the maximum difference between any two values of θ which are both in H and lie on a line ψ = constant. Then

$$\int_H \delta(\psi)\, d\psi$$

is proportional to the area of H. Thus $f(\theta, \phi)$ is proportional to D, the proportionality constant being independent of θ and ϕ.

As for D, it is the absolute value of the vector product of the vector (c, d) by the vector from M' to M. Hence

$$D = T \cdot \text{abs. value of} \begin{vmatrix} (-R\sin\theta + R'\cos\theta)(R\cos\theta + R'\sin\theta) \\ (-k + \cos\psi) \qquad\qquad \sin\psi \end{vmatrix}$$

$$= T \mid D^* \mid$$

where

$$\begin{aligned} D^* &= +R(-\sin\theta\sin\psi - \cos\theta\cos\psi) \\ &\quad +R'(\cos\theta\sin\psi - \sin\theta\cos\psi) \\ &\quad +k(R\cos\theta + R'\sin\theta) \\ &= -R\cos\phi + R'\sin\phi + k(R\cos\theta + R'\sin\theta). \end{aligned}$$

Not all couples (θ, ϕ) may occur as contacts. The relative velocity of particle B with respect to A is

$$-k + \cos\psi,\ \sin\psi.$$

A contact is possible at (θ, ϕ) if and only if the particle B enters the curve $R(\theta)$ from without. Since the curve $R = \omega(\theta)$ is convex by hypothesis, this will occur when and only when the angle, from the tangent to the curve at θ to the relative velocity vector of B is between zero and π. But this is exactly the condition that $D^* > 0$. We conclude that

$$f(\theta, \phi) = c(D^* + \mid D^* \mid),$$

where c is a constant defined by

$$\int_0^{2\pi}\int_0^{2\pi} c(D^* + \mid D^* \mid)\, d\theta\, d\phi = 1.$$

This is the desired result.

Dr. Charles Kittell proposed a somewhat more specialized version of the problem

of this paper to the author, who gave the solution in 1945 while a member of the Statistical Research Group of Columbia University, which was working under contract with the National Defense Research Committee of the Office of Scientific Research and Development.

A COMMENT ON F. N. FRENKIEL'S NOTE "ON THIRD-ORDER CORRELATION AND VORTICITY IN ISOTROPIC TURBULENCE"*

By G. K. BATCHELOR and A. A. TOWNSEND (*Cambridge University*)

The short paper by Mr. F. N. Frenkiel [this Quarterly, **6**, 86-90 (1948)] calls for some comment. Mr. Frenkiel makes certain assumptions about the correlation functions and derives relations for the decay of energy and mean-square vorticity in isotropic turbulence. These relations are not in agreement with measurements which we have already published.[1] Mr. Frenkiel concludes that "if account is taken of probable inaccuracies in the experimental data (in Ref. 1), it appears that the agreement (with his deductions) may be satisfactory after all." In answer to this suggestion, we make the following points.

(1) In our opinion there is no possibility that experimental error would account for the discrepancies between our data and Mr. Frenkiel's results. He finds $d\lambda^2/dt = 7\nu$, whereas according to our measurements, $d\lambda^2/dt = 10\nu$ with a standard deviation of less than 0.4ν. Our measurements were corrected for the effect of finite length of the recording hot-wire but, in any case, the correction leaves $d\lambda^2/dt$ unaltered provided the wire length is small enough to be comparable with λ. Mr. Frenkiel also suggests lack of isotropy as a possible source of error. The presence of isotropy to a sufficient approximation seems to us to have been checked very well by the consistency of measurements of $\overline{u^2}$ and λ (which are related by the energy equations for isotropic turbulence) and of $\overline{u^2}$, λ, $f_0^{iv}\lambda^4$ and $k_0''\lambda$ (which are related by the vorticity equation).

(2) There is a strong body of theory[2] concerning the mechanism of turbulence at high Reynolds numbers, which has been shown[3] to link up very satisfactorily with the measurements of Ref. 1. Any contrary results, such as that given by Mr. Frenkiel for the variation of $k_0'''\lambda^3$ during decay, would therefore need strong experimental and theoretical backing.

(3) Mr. Frenkiel gives no reasons to support his assumption that the correlation curve is self-preserving outside the (small) region in which r is comparable with λ. Indeed, it leads him into inconsistencies. Later in his paper he assumes, in agreement with our measurements, that $f_0^{iv}\lambda^4$ is constant during decay. This implies that the correlation curve is also self-preserving for small values of r and it has now been assumed in effect that $f(r)$ is completely self-preserving. Equation (6) of his paper then leads to the conclusion that $\lambda(\overline{u^2})^{1/2}/\nu$ is constant during decay, whereas his published result is that this quantity varies as $t^{-3/14}$. Again, he is willing to assume $f_0^{iv}\lambda^4$ independent of t, and any physical basis for this assumption would seem to lead also to $k_0'''\lambda^3$ being independent of t (as our measurements show), but Mr. Frenkiel finds $k_0'''\lambda^3$ is proportional to $t^{3/14}$.

*Received June 16, 1948.

[1] G. K. Batchelor and A. A. Townsend, Proc. Roy. Soc. (A) **190**, 534 (1947).

[2] A. N. Kolmogoroff, C. R. Acad. Sci. U.R.S.S. **30**, 301 (1941) and **32**, 16 (1941).

[3] G. K. Batchelor, Proc. Camb. Phil. Soc. **43**, 534 (1947).

A PROBLEM CONCERNING ORTHOGONAL TRAJECTORIES*

By F. HERZOG and C. P. WELLS (*Michigan State College*)

1. Introduction. The purpose of this note is to discuss a class of conjugate harmonic functions u, v which possess a unique geometric property. This property will be defined precisely in Sec. 2. For the present we shall describe it in terms relative to a two-dimensional steady state heat flow problem.

Let S be a given region in a plane which we choose to call the uv-plane, and let C be the boundary of the region. Consider a family of heat flow lines $f(u, v) = c_1$ together with the isothermal lines $g(u, v) = c_2$ in the region S. Suppose we desire to make a plot of a set of curves $f_1 , f_2 , f_3 , \cdots$ from the family $f(u, v) = c_1$ and a set $g_1 , g_2 , g_3 , \cdots$ from the family $g(u, v) = c_2$. For this purpose it is often convenient to choose one of the heat flow lines (or isothermal lines), whose position can be determined in advance from some symmetry property of the boundary C, as a reference line for construction of the remaining curves. This heat flow reference line is divided into segments and the end points of the segments taken as origins for the isothermal lines. Then by a process of numerical integration the isothermal lines and the remaining heat flow lines are plotted for the entire region.[1]

The following problem then arises: Let f_1 be the reference heat flow line and suppose three isothermal lines g_1 , g_2 , g_3 are chosen to intersect f_1 at segments of equal arc length measured along f_1 . Under what conditions will the isothermal lines g_1 , g_2 , g_3 intersect all remaining heat flow lines in S at segments of equal arc length?

In the following sections we shall determine necessary and sufficient conditions for a family of isothermals to have this property and, moreover, determine all such isothermals.

2. Notations and definitions. We shall assume throughout this paper that we deal with functions having continuous derivatives of the second order; whenever a function occurs as a denominator we exclude points at which this function vanishes. A function $\omega(x, y)$ will be called *separable* if $\omega(x, y) = h(x)k(y)$. It is easily shown that $\omega(x, y)$ is separable if and only if $\omega\omega_{xy} - \omega_x\omega_y = 0$, where here, and in the following, subscripts x and y denote partial derivatives.

It seems advantageous to replace the letters c_1 and c_2 of Sec. 1 by x and y, respectively, and to deal with the inverse functions of $f(u, v) = x$ and $g(u, v) = y$. Thus let

$$u = u(x, y), \qquad v = v(x, y) \tag{1}$$

be defined in a region R of the xy-plane. The relations (1) represent a mapping of R on a region S of the uv-plane. We shall denote the curves in S obtained by putting $y =$ const in (1) as the family $\mathfrak{X}$, and the curves obtained by putting $x =$ const as the family $\mathfrak{Y}$. In other words, (1) maps the lines parallel to the x-axis into the family $\mathfrak{X}$ and the lines parallel to the y-axis into the family $\mathfrak{Y}$.

We shall use the usual notation for the three fundamental coefficients $E = u_x^2 + v_x^2 , F = u_x u_y + v_x v_y , G = u_y^2 + v_y^2$. It is obvious that the families $\mathfrak{X}$ and $\mathfrak{Y}$ remain invariant under any substitution $x = \phi(x_1), y = \psi(y_1)$ in (1); moreover, in this manner

*Recieved July 14, 1948.

[1]See, for example, W. H. McAdams, *Heat transmission*, 2nd ed., McGraw-Hill, New York, 1942, pp. 16-17.

all possible representations of the form (1) of the families $\mathfrak{X}$ and $\mathfrak{Y}$ are obtained. If E_1 , F_1 , G_1 are the three fundamental coefficients in terms of the new variables x_1 , y_1 then

$$E_1 = [\phi'(x_1)]^2 E, \qquad F_1 = \phi'(x_1)\psi'(y_1)F, \qquad G_1 = [\psi'(y_1)]^2 G. \tag{2}$$

The necessary and sufficient condition for orthogonality of $\mathfrak{X}$ and $\mathfrak{Y}$ is $F = 0$. Two families of orthogonal trajectories $\mathfrak{X}$ and $\mathfrak{Y}$ are called *isothermic trajectories* or *isothermals* if it is possible to make substitutions $x = \phi(x_1)$ and $y = \psi(y_1)$ in (1) such that u and v become conjugate harmonic functions in x_1 and y_1 , i.e., such that $w = u + iv$ becomes an analytic function of $z_1 = x_1 + iy_1$. As is well-known, two families of orthogonal trajectories $\mathfrak{X}$ and $\mathfrak{Y}$, given by (1), are isothermic if and only if E/G is separable.[2]

We shall now define for orthogonal trajectories the property which is described in Sec. 1 and which is the fundamental concept of this paper. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two families of orthogonal trajectories. We shall say that one of them, say $\mathfrak{X}$, is *of proportional arc length* if the following condition is satisfied: Let Y_1 , Y_2 , Y_3 be any three curves of $\mathfrak{Y}$, and let s_1 be the arc length of any curve of $\mathfrak{X}$ between Y_1 and Y_2 and s_2 be the arc length of the same curve between Y_2 and Y_3 . Then the ratio s_1/s_2 is to be constant for all curves of $\mathfrak{X}$, i.e., dependent only on the choice of Y_1 , Y_2 , Y_3 . It is obvious that we might express this property of $\mathfrak{X}$ also in the following way: Any three trajectories of $\mathfrak{Y}$ which cut out segments of equal arc length from one curve of $\mathfrak{X}$ do the same for any other curve of $\mathfrak{X}$. In an analogous manner we define what is meant for $\mathfrak{Y}$ to be of proportional arc length.

3. A condition for proportional arc length. We shall prove the following theorem, which gives a necessary and sufficient condition for proportional arc length.

THEOREM 1. *If $\mathfrak{X}$ and $\mathfrak{Y}$, defined by* (1), *are orthogonal trajectories, then $\mathfrak{X}$ is of proportional arc length if and only if $E = u_x^2 + v_x^2$ is separable.*[3] *Similarly, $\mathfrak{Y}$ is of proportional arc length if and only if $G = u_y^2 + v_y^2$ is separable.*

Let three trajectories of $\mathfrak{Y}$, corresponding to the values $x = x_0$, $x = x_1$, $x = x_2$, be given and consider any two trajectories of $\mathfrak{X}$, corresponding to the values $y = y_1$ and $y = y_2$. The element of arc length along any curve of $\mathfrak{X}$ is given by $ds = q(x, y)\,dx$, where we write $q(x, y) = [E(x, y)]^{1/2} = (u_x^2 + v_x^2)^{1/2}$. If $\mathfrak{X}$ is of proportional arc length we have, by the definition given above (see Sec. 2):

$$\int_{x_0}^{x_1} q(x, y_1)\,dx : \int_{x_0}^{x_2} q(x, y_1)\,dx = \int_{x_0}^{x_1} q(x, y_2)\,dx : \int_{x_0}^{x_2} q(x, y_2)\,dx; \tag{3}$$

this has to hold for all values of x_0 , x_1 , x_2 and y_1 , y_2 , restricted only by the region of definition R. We conclude that the function

$$\int_{x_0}^{x_1} q(x, y)\,dx \Big/ \int_{x_0}^{x_2} q(x, y)\,dx \tag{4}$$

is independent of y. If we consider x_0 and x_2 as fixed, then (4) becomes a function of x_1 alone, which we shall call $h(x_1)$. The denominator of (4) is a function of y alone, say $k(y)$. Hence we have

[2] See G. Scheffers, *Anwendung der Differential- und Integralrechnung auf Geometrie*, Walter de Gruyter, Berlin und Leipzig, 1923, vol. 1, 3rd ed., p. 170.

[3] The separability of E is independent of the particular representation (1) of $\mathfrak{X}$ and $\mathfrak{Y}$. This can be seen directly from (2).

$$\int_{x_0}^{x_1} q(x, y)\, dx = h(x_1)k(y). \tag{5}$$

Differentiating with respect to x_1 and replacing x_1 by x, we obtain $q(x, y) = h'(x)k(y)$. Thus $E = q^2$ is separable and the necessity of the condition is proved. To prove its sufficiency we note that (5) follows at once from the separability of E and that (5) immediately implies the proportion (3). This completes the proof of Th. 1.

The question at once arises as to whether there are families of orthogonal trajectories in which $\mathfrak{X}$ is of proportional arc length but $\mathfrak{Y}$ is not. We shall see below (Th. 2) that this can only occur if $\mathfrak{X}$ and $\mathfrak{Y}$ are non-isothermic. In this case the conditions

$$E \text{ separable}, \qquad F = 0, \qquad G \text{ non-separable}, \tag{6}$$

are compatible. We shall give two examples of such trajectories.

I. Let $u = y + e^{x-y}$, $v = e^{(x-y)/2}(1 - e^{x-y})^{1/2} + \sin^{-1} e^{(x-y)/2}$. The region R is taken as the half-plane above the line $y = x$, so that $0 < e^{x-y} < 1$ for (x, y) in R. It is easily verified that $E = e^{x-y}$, $F = 0$, $G = 1 - e^{x-y}$, and hence (6) is satisfied.

II. Let $p(x)$, $q(y)$ and $r(y)$, defined for $a < x < b$ and $c < y < d$, respectively, be such that neither $p(x)$ nor $q(y)$ nor $r(y)/q'(y)$ is constant. Let

$$u = p(x) \cos q(y) - \int r(y) \sin q(y)\, dy,$$

$$v = p(x) \sin q(y) + \int r(y) \cos q(y)\, dy.$$

The region R is the rectangle $a < x < b$, $c < y < d$. We obtain $E = [p'(x)]^2$, $F = 0$, and $G = [p(x)q'(y) + r(y)]^2$. In order to show that (6) is satisfied, we only have to convince ourselves that G is not separable. We apply the separability test (see Sec. 2) to $\omega(x, y) = p(x)q'(y) + r(y)$ and obtain $\omega\omega_{xy} - \omega_x\omega_y = -p'(x)[q'(y)]^2\,(d/dy)[r(y)/q'(y)]$, which is different from zero by the conditions mentioned at the beginning of this example.

We remark that in the second example E is actually a function of x alone. The geometric interpretation of this special situation is that the family $\mathfrak{X}$ is not only of proportional arc length but, as we might say, of *equal* arc length, i.e., any two fixed curves of $\mathfrak{Y}$ cut out a segment of the same arc length from all curves of $\mathfrak{X}$.

4. Isothermals. Before taking up the main problem of this paper (see Sec. 1), we prove the following theorem.

THEOREM 2. *If two families of orthogonal trajectories are both of proportional arc length then they are isothermals. Conversely, if one of two families of isothermals is of proportional arc length so is the other.*

If two of the three functions E, G, and E/G are separable, so is the third. Hence Th. 2 follows at once from Th. 1 and from the fact (see Sec. 2) that the separability of E/G is a sufficient and necessary condition for isothermals.

The second part of Th. 2 allows us to speak of "isothermals of proportional arc length", without our having to mention the families $\mathfrak{X}$ or $\mathfrak{Y}$ specifically. Our aim is to find the totality of all isothermals of proportional arc length. In preparation of the solution of this problem we shall first make the following remarks.

In the first place, we may assume right from the start that the functions u and v in (1), which define our isothermals, are real and imaginary parts, respectively, of an

analytic function $w = F(z)$, where $z = x + iy$ (see Sec. 2). Secondly, it is obvious that a substitution of the form $z = \alpha z_1 + \beta$, $\alpha \neq 0$, will leave the two families of isothermals invariant if α is real, and that it will merely interchange the two families with one another if α is purely imaginary. Thirdly, we shall say that one pair of families of isothermals is *of the same type* as another pair if the second pair can be obtained from the first one by a rotation, expansion (contraction) and translation of the w-plane, i.e., by a transformation $w = aw_1 + b$, where a is any complex number different from zero. To summarize, we shall consider two pairs of families of isothermals to be of the same type if the analytic functions $w = F(z)$ and $w_1 = G(z_1)$ defining them are related to one another in the following manner:

$$w = aw_1 + b, \qquad a \neq 0,$$
$$z = \alpha z_1 + \beta, \qquad \alpha \neq 0, \qquad \alpha \text{ real or purely imaginary}. \tag{7}$$

We are now ready to state the principle result of our paper.

THEOREM 3. *The totality of different types of isothermals of proportional arc length are those obtained from one of the following functions*: (i) $w = z$, (ii) $w = e^z$, (iii) $w = \exp(ze^{i\gamma})$, *where* $0 < \gamma < \pi/2$, *and* (iv)

$$w = \int_0^z \exp(-\zeta^2)\, d\zeta.$$

Let $w = F(z) = u(x, y) + iv(x, y)$ be such that the isothermals defined by $F(z)$ are of proportional arc length. Since $E = G = |F'(z)|^2$ we conclude from Th. 1 that $|F'(z)|$ is separable and hence $\log|F'(z)| = M(x) + N(y)$. Now $\log|F'(z)|$ is a harmonic function and, therefore, $M''(x) + N''(y) = 0$ so that $M''(x) = -N''(y) = 2A$, where A is a real constant. We thus obtain $M(x) = Ax^2 + B_1x + C_1$ and $N(y) = -Ay^2 + B_2y + C_2$ with real B_1, B_2, C_1, C_2. Thus $\log|F'(z)| = A(x^2 - y^2) + B_1x + B_2y + C_3$, where $C_3 = C_1 + C_2$, and hence $\log F'(z) = Az^2 + Bz + C$, where $B = B_1 - iB_2$ and C is any complex number whose real part equals C_3. We thus obtain $F'(z) = \exp(Az^2 + Bz + C)$ with real A and distinguish between the following cases.

(i) $A = 0$, $B = 0$. We obtain $w = F(z) = mz + p$ with $m = e^C$. The substitutions (7) with $w = mw_1 + p$, $z = z_1$ give $w_1 = z_1$. Thus all isothermals of Case (i) are of the same type, namely, $w = z$. The families $\mathfrak{X}$ and $\mathfrak{Y}$ consist of lines parallel to the real and imaginary axes, respectively. Obviously, both families are trajectories of equal arc length, in the sense defined above at the end of Sec. 3.

(ii) $A = 0$, $B \neq 0$, B real or purely imaginary. We obtain $w = F(z) = me^{Bz} + p$, where $m = B^{-1}e^C$. The substitutions (7) with $w = mw_1 + p$, $z_1 = Bz$ give $w_1 = \exp(z_1)$. Thus all isothermals of Case (ii) are of the same type, namely, $w = e^z$. The family $\mathfrak{X}$ consists of rays from the origin to the point at infinity, the family $\mathfrak{Y}$ of circles about the origin. The family $\mathfrak{X}$ is again of equal arc length but the family $\mathfrak{Y}$ is not.

(iii) $A = 0$, B neither real nor purely imaginary. As in Case (ii) we obtain $w = F(z) = me^{Bz} + p$ with $m = B^{-1}e^C$. We now write $B = |B| \exp[i(\gamma + \pi n/2)]$, where n is an integer and $0 < \gamma < \pi/2$. The substitutions (7) with $w = mw_1 + p$, $z_1 = |B| z \exp(i\pi n/2)$ give $w_1 = \exp(z_1 e^{i\gamma})$. Therefore, every pair of isothermals of Case (iii) is of the same type as $w = \exp(ze^{i\gamma})$ for an appropriate value of γ, $0 < \gamma < \pi/2$. The families $\mathfrak{X}$ and $\mathfrak{Y}$ may be written in polar coordinates r, θ as follows:

$$\mathfrak{X}: r = \exp(\theta \cot \gamma - y \csc \gamma), \qquad y = \text{const},$$

$$\mathfrak{Y}: r = \exp(-\theta \tan \gamma + x \sec \gamma), \qquad x = \text{const}.$$

Each of these families consists of logarithmic spirals about the origin; the curves of $\mathfrak{X}$ intersect the rays through the origin at the angle γ and those of $\mathfrak{Y}$ at the angle $\gamma - \pi/2$. Thus for two different values of γ, $0 < \gamma < \pi/2$, two different types of isothermals are obtained.[4]

(iv) $A \neq 0$. We put $A = -\nu^2$, so that ν is real or purely imaginary since A is real. We write

$$w = F(z) = \int_{B/2\nu^2}^{z} \exp(-\nu^2\zeta^2 + B\zeta + C)\, d\zeta + p.$$

The substitutions (7) with $w = p + (w_1/\nu) \exp(\Delta/4\nu^2)$, $z_1 = \nu z - B/2\nu$, where $\Delta = B^2 + 4\nu^2 C$, give

$$w_1 = \int_0^{z_1} \exp(-\zeta^2)\, d\zeta.$$

Thus all isothermals of Case (iv) are of the same type, namely,

$$w = \int_0^z \exp(-\zeta^2)\, d\zeta = \operatorname{Erf}(z).$$

We shall not attempt here to describe these isothermals in detail. We merely restrict ourselves to stating that the curves of $\mathfrak{X}$ approach $\pm(\pi)^{1/2}/2$ as $x \to \pm\infty$, while the curves of $\mathfrak{Y}$ approach the point at infinity as $y \to \pm\infty$. A brief discussion of the complex error function is given by Whittaker and Watson;[5] a more detailed discussion can be found in a recent monograph by Rosser.[6]

5. Physical interpretations. We return to the notation of Sec. 1 and assume that $f(u, v) = c_1$ are the heat flow lines and $g(u, v) = c_2$ are the isothermal lines of a two-dimensional steady state heat flow problem. Let F be the flux of heat across any isothermal line g_i. The flux will in general vary from point to point along g_i. Let F_1 be the flux across g_i in the direction of f_1, F_2 in the direction of f_2, etc. Then for two neighboring isothermal lines g_1 and g_2, defined by $g(u, v) = c$ and $g(u, v) = c + \Delta c$, respectively, where Δc is small, the value of the flux is given approximately by

$$F_1 = -k \frac{\Delta c}{\Delta n_1}, \qquad F_2 = -k \frac{\Delta c}{\Delta n_2},$$

where k is the thermal conductivity and where Δn_i represents the distance between the curves g_1 and g_2, measured along the curves f_i, $i = 1, 2$. Now suppose the isothermals are of proportional arc length. Then the ratio F_1/F_2 will be the same for all isothermals g_i.

[4]For an example in fluid dynamics involving these isothermals, see G. Hamel, *Spiralförmige Bewegungen zäher Flüssigkeiten*, Jber. Deutschen Math. Verein. **25**, 34-60 (1917).

[5]E. T. Whittaker and G. N. Watson, *Modern analysis*, Macmillan and Co., New York, 1947, p. 341.

[6]J. B. Rosser, *Theory and Application of*

$$\int_0^z \exp(-x^2)\, dx \quad \text{and} \quad \int_0^z \exp(-p^2y^2)\, dy \int_0^y \exp(-x^2)\, dx,$$

Mapleton House, Brooklyn, 1948.

Obviously, the problem as described above in terms of flux of heat can be restated in terms of other physical quantities such as potential in an electrostatic field, velocity potential in fluid flow, gravitational potential, etc.

For example, consider a potential function $f(u, v) = c$ and a vector force function $\mathbf{F}(u, v)$. Let $f(u, v) = c$ and $f(u, v) = c + \Delta c$ be two neighboring equipotential curves. Then $\mathbf{F}(u, v)$ acts in a direction orthogonal to the equipotential curves and its magnitude is given by $F = \Delta c/\Delta n$, where Δn denotes the distance between the two curves. That is, the magnitude of $\mathbf{F}$ is inversely proportional to the distance between the two curves. Hence if $f_1, f_2, f_3, \cdots$ are a set of equipotential curves, and g_1, g_2 any two lines of force and if $\mathbf{F}_1$ acts along g_1, $\mathbf{F}_2$ along g_2 the property of proportional arc length leads to $F_1/F_2 = \text{const.}$

BOOK REVIEWS

Mathematics our great heritage. Selected and Edited by William L. Schaaf, Ph.D. Harper & Brothers, New York. xi + 291 pp. $3.50.

This book is a collection of essays on mathematics and its relationships to other fields of endeavor, including the fine arts, philosophy, experimental science and technology. The authors include both eminent mathematicians and those whose primary fields are not pure mathematics, but whose work brings them in close contact with mathematics, and has induced them to examine the relations between mathematics and other activities. The essays are on a high level, and should be of particular interest to applied mathematicians.

P. S. Symonds

The principles of quantum mechanics. By P. A. M. Dirac. Oxford, at the Clarendon Press, 1947. xii + 311 pp. $9.00.

This work, which has always been a cornerstone in the theoretical foundation of Quantum Mechanics, retains all of its stature in this new edition. Substantially the same subject matter is covered as in the two earlier editions. However, it is apparent that the new edition has been very largely rewritten. The author has changed his notation, but the new notation is almost self-explanatory and should cause little difficulty. The new "bra" and "ket" vector notation for states allows a more direct connection to be made between the abstract algebra of states and observables, and the theory of representations. At the same time, the author has put the formal algebra of states and observables into neater and more elegant form.

Other significant changes are in the presentation of the theory of systems containing identical particles, which is in a somewhat simpler form in this new edition, and in the treatment of quantum electrodynamics, which has been carried a little further than in the previous edition. However, the problem of divergent solutions remains, and the theory cannot be carried to a satisfactory conclusion.

D. F. Hornig

Mechanical behavior of high polymers. By Turner Alfrey, Jr. Interscience Publishers, Inc., New York, 1948. xiv + 580 pp. $9.50.

While the number of books dealing with deformation and strength of materials considered as an elastic continuum is growing like mushrooms, no book has so far been written in which the mechanical behavior of real materials (which, being formed by aggregation of particles into groups, are neither

homogeneous nor isotropic) is being discussed from both the phenomenological and the structural point of view, and the observable phenomenological behavior related to the underlying structural mechanism. Therefore the appearance of a treatise covering a very complex and a very important section of this general subject of mechanical behavior of real materials—the field of high polymers—is most welcome. The author's principal purpose is clear from the preface in which he states that "the analysis of the mechanical behavior has been made on two levels—a phenomenological description and a molecular interpretation."

The book is divided into six main chapters: A. Introduction; B. Plastoelastic Behavior of Amorphous Linear High Polymers; C. Three-Dimensional Cross-Linked Polymers; D. Crystallization of High Polymers; E. Systems Containing High Polymers and Materials of Low Molecular Weight; F. Ultimate Strength and Related Properties. The text is followed by four Appendici, the first of which gives a very short explanation of tensor representation, the next two deal with mathematical concepts used in the analysis of viscoelastic bodies, while the last contains a short discussion of mechanical testing.

The introductory chapter presents the basic concepts of the mechanics of deformable solid and of the viscous and the complex (non-Newtonian) liquid, limited to conditions of equilibrium and of stationary flow. This is followed by a discussion of the structural (molecular) mechanisms of flow and of viscoelastic deformation, illustrated by simple mechanical models consisting of springs and dashpots. The discussion of stress, strain and flow starts on a level which is rather elementary; very soon, however, tensor notation is introduced together with a somewhat erratic discussion of crystal geometry in relation to the multiple-constant theory of elasticity, and of the compatibility conditions for infinitesimal strain; these concepts are hardly very important for an understanding of the main subject. On the other hand, one of the major problems of the mechanics of highly deformable bodies, such as many high polymers, the arbitrariness of the definition of strain and the analysis of finite deformations and strains, is not considered worthy of any, but the most superficial treatment; while Murnaghan's analysis is given a few words, there is no mention of Hencky's or Seth's work, although their methods have been applied with success to the representation of the deformation of rubber. The discussion of flow, which follows, is mainly concerned with the presentation of the equations of the capillary and the rotating cylinder viscometers for the Newtonian fluid and the Bingham body, as developed by Buckingham and Reiner. The discussion of thixotropy lacks the fundamental approach to be found in Freundlich's work to which it refers, and is more a discussion of specific experimental procedure; there is no attempt to present yield-limit, thixotropy and non-Newtonian flow within an integrated structural picture. The rest of the chapter is taken up by a simple analysis of the Maxwell and the Voigt (Kelvin) model-unit, by a discussion of the bulk-modulus of single crystals in terms of atomic structure, and by a presentation of the structural mechanism of viscous liquid flow.

The second chapter treats the inelastic behavior of amorphous, linear high polymers, which are the thermoplastics and the unvulcanized rubbers. The molecular structure is broadly discussed and its response to load quantitatively reproduced with the aid of combined linear mechanical models. In selecting the type of model, the author expresses a definite preference for the combination in series of "retarded" Voigt (Kelvin)-units, as against the parallel combination of "relaxing" (Maxwell)-units. The discussion of the dynamics of viscoelastic behavior introduces methods of circuit analysis, which are simple and very effective in the analysis of complex models.

The analysis of the effect of non-homogeneous and combined stresses starts very promising; an attempt is made to derive the equations for the general conditions of stress for a Maxwell body, from which relations between the components of the velocity of relaxation and of the stress tensor could be derived, but is not followed up. The presentation of the elastic-viscoelastic analogy, however, is of importance; its understanding requires a study of Appendix III, reproduction of an original paper by the author.

The engineer interested in photoelasticity, who may turn to the book for fundamental information concerning the interrelation of photoelastic and mechanical behavior of the principal photoelastic materials will be disappointed. The discussion of the photoelastic effects in high polymers is more or less on the level found in most textbooks on photoelasticity.

The third chapter contains a discussion of the behavior of three-dimensional, cross-linked polymers, such as thermosetting plastics and vulcanized rubbers. Mechanical model analysis can no longer be applied, because of the cross-linking of molecular chains. The network analysis used instead in the correlation of phenomenological and structural response to load is based on concepts of statistical mechanics. Nearly one-half of this chapter is taken up by verbatim quotations of a few papers, which are easily accessible.

The fourth chapter deals with the crystallization of high polymers. The discussion of the complex phenomenon of "crystallization" and of the mechanical properties of "crystalline" (read "anisotropic") polymers, which comprise all fibrous materials, yarns and filaments, is thorough, although no serious attempt at a correlation of phenomenological behavior and structural response is made beyond the level of a purely descriptive experimental approach.

The fifth chapter brings an analysis of the behavior of polymers admixed with components of low molecular weight. Again, the high complexity of the structure makes any quantitative correlation between large-scale behavior and structure impracticable, and the discussion is largely a qualitative interpretation of experiments. A short discussion of the viscosity of suspensions is included.

In the comparatively short sixth chapter an attempt is made to explain fracture in terms of the structural mechanism involved. However, the discussion of the phenomenological aspect is oversimplified. Even the important aspect of the different significance, with regard to failure by excessive deformation and by fracture, of the tensors of volumetric and of deviatoric stress is omitted. Its consideration would immediately have shown up the untenability of the proposed concept of a "single limiting (failure) stress surface, containing a region of fracture and a region of plastic failure" (p. 480). No mention is made of the interrelation between fracture and the inelastic deformation preceding it. Since it is this interrelation which is at the root of the time-dependence of fracture-stress, the discussion of this dependence must necessarily remain on a purely empirical level. It is but in the sections dealing with the structural aspects, particularly the dependence of tensile strength on molecular weight and orientation, in which the principal object of the book is again brought into sharp focus. All chapters are followed by extensive lists of references.

Appendix II, a reproduction of a paper by the author and P. M. Doty is an excellent comparison of the different mathematical methods used to represent visco-elastic behavior.

The author, in attempting to write a comprehensive book on a subject which is not only very complex, but in the flux of rapid development, has taken upon himself an extremely difficult task for which the large number of workers in high polymer research owe him a debt of gratitude The principal weakness of the book is the lack of balance in the presentation of the various aspects of the subject, particularly the phenomenological (mechanical) aspect. The omission of an adequate discussion of the thermodynamic foundation of mechanical behavior is a definite handicap for the reader with an engineering background. A second weakness is the sometimes rather loose terminology and the lack of definition of unconventional terms used, such as "inertial elasticity", "plastoelastic", "stress biased molecular diffusion" (which is used as a synonym for Brownian motion), etc.

Because of its unique character the book will easily find its way to the desk of most workers in polymer research.

A. M. FREUDENTHAL

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In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u\, du \text{ is preferable to } \int \frac{\sin u}{u}\, du.$$

Sometimes the intended grouping of symbols in handwritten formulas can be made clear by slight variations in the size of parentheses; this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t\,(a + bx).$$

In complicated formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{[a + (b + cx)^n] \cos ky\}^2 \text{ is preferable to } ([a + (b + cx)^n) \cos ky)^2.$$

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